Lecture 12

- Singular ~~Approx~~ Perturbation Method
- WKB approximation
- Asymptotic approximation of some integrals

Singular Perturbation

- Failure of Regular perturbation

Consider the following example

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0 \,, \qquad 0 < t < 1$$
$$0 < \varepsilon \ll 1$$

$$y(0) = 0$$
$$y(1) = 1$$

assume a perturbation series

$$y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$

$$\varepsilon \left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \dots \right) + y_0' + \varepsilon y_1' + \varepsilon^2 y_2'$$
$$+ \varepsilon \left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \dots \right) + y_0 + \varepsilon y_1 + \dots = 0$$

ε^0 $$y_0' + y_0 = 0 \qquad y_0(0) = 0 \,,\ y_0(1) = 1$$

$$y_0'' + y_1' + y_0' + y_1 = 0 \qquad y_1(0) = 0 \quad y_1(0) = 0$$

$$y_0 = c\,e^{-t}$$

one const.

$$y_0(0) = c = 0$$

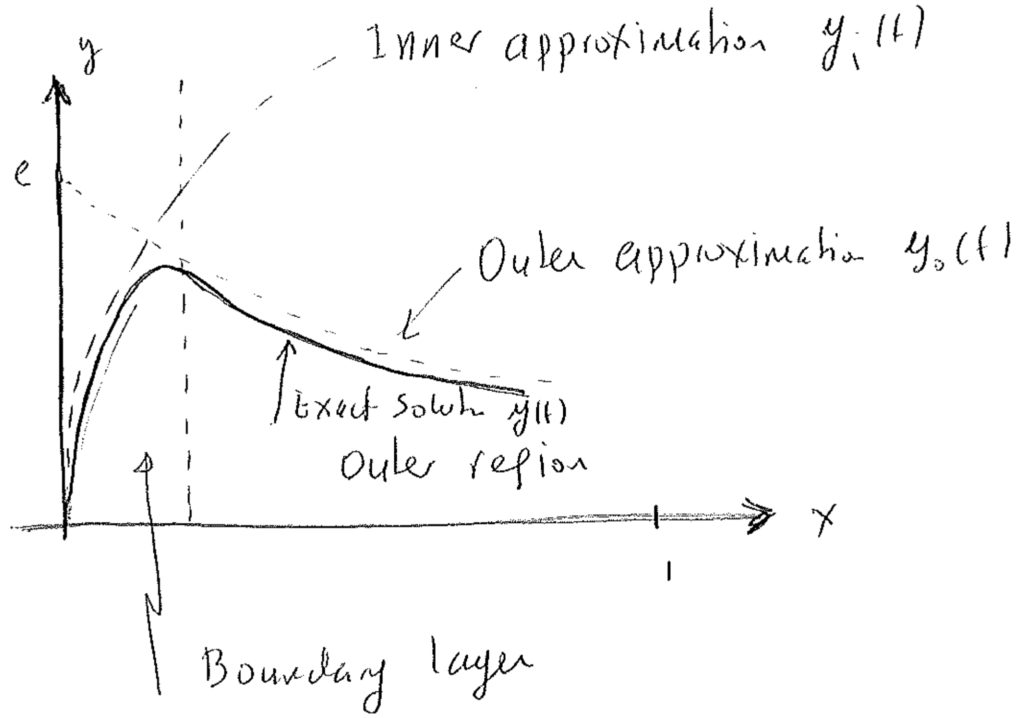
NOT CONSISTENT

Inner and Outer approximation

$\varepsilon = 0$ $\quad y' + y = 0 \quad$ $y(0) = 0$, $y(1) = 1$

- differentiability order reduces to 1.
- exact solution is possible

$$y(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}}\left(e^{-t} - e^{-t/\varepsilon}\right)$$



Schematic of exact solution compared to inner and outer approximations for a fixed value of ε $\quad t_0 = \dfrac{\varepsilon \ln \varepsilon}{\varepsilon - 1}$

$$y(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-t} - e^{-t/\varepsilon} \right)$$

$t > 1$

$$y(t) = \frac{1}{1 - e^{1-1/\varepsilon}} \left(e^{1-t} - e^{1-t/\varepsilon} \right)$$

for small ε

$$e^{-1} - e^{-1/\varepsilon} \sim e^{-1}$$

$$y(t) \simeq e^{1-t} - e^{1-t/\varepsilon}$$

for $t \sim 1$ $\quad \varepsilon \to 0$

$$y(t) \simeq e^{1-t} \qquad t = O(1)$$

$$y' = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(-e^{-t} + \frac{1}{\varepsilon} e^{-t/\varepsilon} \right)$$

$$y'' = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-t} - \frac{1}{\varepsilon^2} e^{-t/\varepsilon} \right)$$

1) • suppose ε is small
t is in the narrow
boundary layer near $t=0$.

• $y''(\varepsilon) = O(\varepsilon^{-2})$
hence $\varepsilon y''$ is not small $\qquad \varepsilon y'' = O(\frac{1}{\varepsilon})$

2) In the outer region

$$y''\left(\tfrac{1}{2}\right) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-1/2} - \frac{1}{\varepsilon^2} e^{-1/2\varepsilon} \right)$$

$$= O(1)$$

$\varepsilon y''$ is small

for $t \sim 0$

$$y(t) \approx e - e^{1-t/\varepsilon}$$

hence

$$y_i(t) = e - e^{1-t/\varepsilon} \qquad t \text{ small}$$

$$y_0(t) = e^{1-t} \qquad t = O(1)$$

$t \sim \sqrt{\varepsilon}$

matching

as $\varepsilon \to 0$

Singular Perturbations

1. When the small parameter multiplies the highest derivative in the problem

$\varepsilon u'' + \dots$

2. When setting the parameter equal to zero completely changes the character of the problem

   a) In the case of PDE the character of the problem changes, hyperbolic reduces to parabolic $u_{tt} - \varepsilon u_{xx} = f(u, u_x, \dots)$

   $\varepsilon \to 0 \quad u_{tt} = f$

   b) Changes the order of DE

   c) Changes the order of algebraic eqn

   $\varepsilon x^2 + 2x + 1 = 0$

3. When problem occur on infinite domain

4. When singular points are present in the interval of interest

5. Physical model has time-variables more than are.

Boundary Layer Analysis

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0 \qquad 0 < t < 1$$
$$0 < \varepsilon << 1$$

$$y(0) = 0, \quad y(1) = 1.$$

Outer solution $t = O(1)$

$\varepsilon y''$ is small. $F(t, y, \dot{y}, \ddot{y}, \varepsilon = 0) = 0$ is consistent

$$y' + y = 0$$

$$y = ce^{-t}$$

$$y(1) = 1 \Rightarrow ce^{-1} = 1 \qquad c = e$$

$$y = e^{1-t} \qquad t = O(1)$$

inner soln.

$t \sim \varepsilon$ $\varepsilon y''$ is not small anymore new scaling

$$\tau = \frac{t}{\delta(\varepsilon)} \quad \Rightarrow \quad t = \delta(\varepsilon)\,\tau$$

$\varepsilon \to 0 \quad \delta \to 0$

$$\frac{dy}{dt} = \frac{dy}{d\tau}\,\frac{1}{\delta} \quad \Rightarrow$$

$$\frac{\varepsilon}{\delta^2(\varepsilon)}\,\frac{d^2y}{d\tau^2} + \frac{1}{\delta(\varepsilon)}\,\frac{dy}{d\tau} + \frac{\varepsilon}{\delta(\varepsilon)}\,\frac{dy}{d\tau} + y = 0$$

Comparison of terms

i) $\dfrac{\varepsilon}{\delta^2}\,, \quad \dfrac{1}{\delta}$ are of the same order (larger terms and the other may be neglected)

ii) $\dfrac{\varepsilon}{\delta^2}\,, \quad \dfrac{\varepsilon}{\delta}$ are of the same order

iii) $\dfrac{\varepsilon}{\delta^2} \quad 1$ - - - -

only the first comparison is consistent

i) $\delta \sim \varepsilon$

$$\frac{1}{\varepsilon} \qquad \frac{1}{\varepsilon} \qquad 1 \qquad 1$$

ii) $\delta \sim 1$ original problem

iii) $\delta \sim \sqrt{\varepsilon}$

$$1 \qquad \frac{1}{\sqrt{\varepsilon}} \qquad \frac{1}{\sqrt{\varepsilon}} \qquad 1$$

not small

larger

hence we choose

$\tau = \dfrac{t}{\varepsilon}$ and DE become

$$\frac{1}{\varepsilon}\frac{d^2y}{d\tau^2} + \frac{1}{\varepsilon}(1+\varepsilon)\frac{dy}{d\tau} + y = 0$$

$$\frac{d^2y}{d\tau^2} + (1+\varepsilon)\frac{dy}{d\tau} + \varepsilon y = 0$$

$$F(\tau, y, \dot{y}, \ddot{y}, 0) = 0$$

$$y'' + y' = 0$$

$$y' = a e^{-\tau}, \quad y = -a e^{-\tau} + b$$

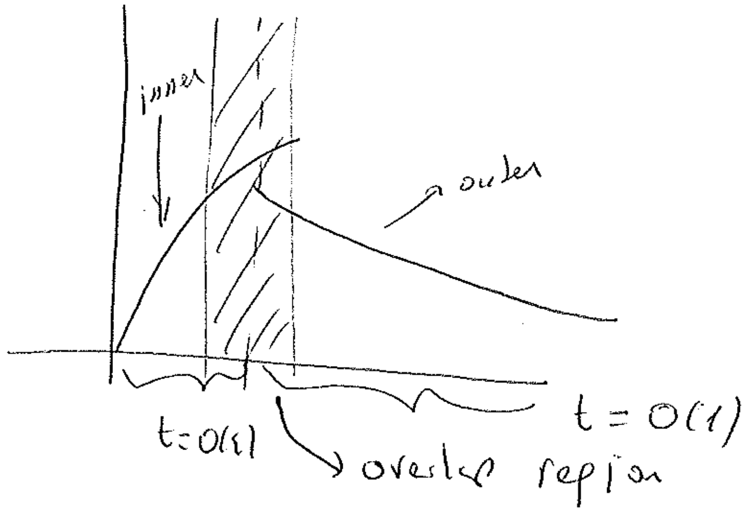
$y(0) = 0 \qquad b = a$

$$y(\tau) = a(1 - e^{-\tau})$$

$$y(t) = a(1 - e^{-t/\varepsilon}) \qquad t = O(\varepsilon)$$

$$y(t) = e^{1-t} \qquad t = O(1)$$

Matching



n positive integer

$$y_i = e^{1-\varepsilon^{1/n}\eta} \qquad t = \varepsilon^{1/n}\eta$$

$$y_o = a\left(1 - e^{-\varepsilon^{\frac{1}{n}-1} t}\right)$$

$$\lim_{\varepsilon \to 0} y_i = e \quad , \quad \lim_{\varepsilon \to 0} y_o = a$$

$$a = e.$$

Uniform Approximation

$$y_0 + y_1 = e^{1-t} + e - e^{1-t/\varepsilon}$$

$$= \begin{cases} e^{1-t} + e & t \neq O(1) \\ 2e - e^{1-t/\varepsilon} & t = O(\varepsilon) \end{cases}$$

One e is extra hence we have to substract the e value.

$$y_u = y_0 + y_1 - e$$
$$= e^{1-t} - e^{1-t/\varepsilon}$$

- y_u satisfies the D.E exactly
- $y_u(0) = 0$

$$y_u(1) = 1 - e^{1-1/\varepsilon} = 1 + O(\varepsilon^n)$$

Right B.C is not satisfied identically approximately

$$\lim \frac{e^{1-1/\varepsilon}}{\varepsilon^n} = 0 \quad \text{for all } n$$

uniform approx

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Singular Perturbations

Problem: Consider the boundary problem

$$\left.\begin{aligned} &\varepsilon y'' + (1+\varepsilon) y' + y = 0 \qquad 0<t<1 \\ &\qquad\qquad\qquad\qquad\qquad 0<\varepsilon \ll 1 \\ &y(0)=0, \quad y(1)=1. \end{aligned}\right\} \quad (1)$$

(.) **Exact solution** is possible

$$y(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-t} - e^{-t/\varepsilon} \right) \quad \checkmark \qquad (2)$$

Solution about $t \sim 0$

$$y'(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(-e^{-t} + \frac{1}{\varepsilon} e^{-t/\varepsilon} \right) \qquad (3)$$

$$y''(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-t} - \frac{1}{\varepsilon^2} e^{-t/\varepsilon} \right) \qquad (4)$$

$$\varepsilon y''(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(\varepsilon e^{-t} - \frac{1}{\varepsilon} e^{-t/\varepsilon} \right) \qquad (5)$$

about $t \sim \varepsilon$ we have

$$\varepsilon y''(t) = O(\varepsilon^{-1}) \qquad (6)$$

hence $\varepsilon y''$ is not allways small

(2) regular perturbation

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0 \quad , \quad y(0) = 0, \quad y(1) = 1$$

$$y_0' + y_0 = 0 \qquad y_0(0) = 0, \quad y_0(1) = 1.$$

$$y_1' + y_1 = -y_0'' - y_0' \quad , \quad y_1(0) = 0, \quad y_1(1) = 0$$

$y_0(t) = c e^{-t}$, $y_0(0) = 0 \Rightarrow c = 0.$

$y_0(t) = 0$ hence $y_0(1) = 1$ not satisfied.

second B.C. $y_0(1) = 1 \Rightarrow c = e^{+1} \Rightarrow$

$y_0(t) = e^{1-t}$

hence

$y_0(t) = 0$ $\quad y_0(0) = 0$ but NOT $y_0(1) = 1$

$y_0(t) = e^{1-t}$ $\quad y_0(1) = 1$ but NOT $y_0(0) = 0$

regular perturbation fails

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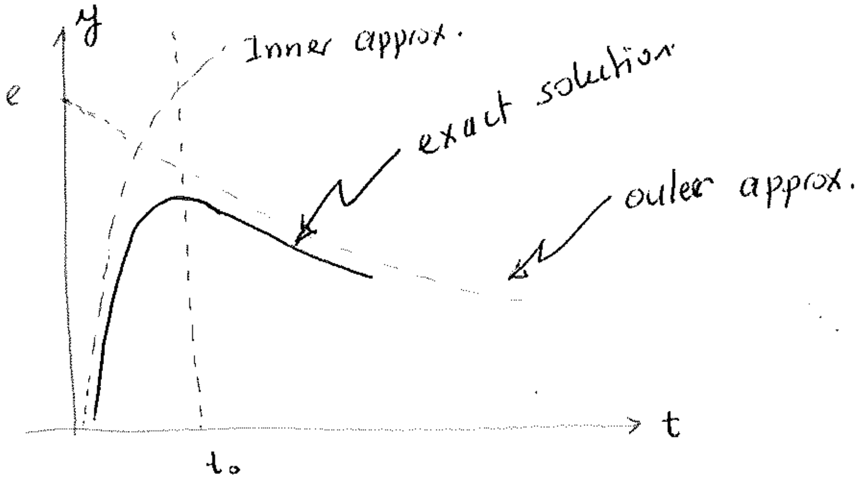
(3) graph of the exact solution



$$t_0 = \frac{\varepsilon \ln \varepsilon}{\varepsilon - 1}$$

$y'(t) = 0$

$$e^{-t} - \frac{1}{\varepsilon} e^{-t/\varepsilon} = 0$$

for ε small

$$e^{-1} - e^{-1/\varepsilon} \approx e^{-1}$$

$$-t_0 = -\ln \varepsilon - \frac{t_0}{\varepsilon_0}$$

a) when $t \sim 1$ then

$$y_0 \approx e^{+1}(e^{-t}) = e^{1-t}$$

$$\left(\frac{1}{\varepsilon_0} - 1\right) t_0 = -\ln \varepsilon$$

$$t_0 = \frac{\varepsilon \ln \varepsilon}{\varepsilon - 1}$$

b) $t \sim 0$ then

$$\lim t_0 = \frac{\ln \varepsilon / \varepsilon^{-1}}{\varepsilon - 1}$$

$$y_i \approx e(1 - e^{-t/\varepsilon}) = e - e^{1 - t/\varepsilon}$$

t small

$$= -\frac{1}{\varepsilon} \varepsilon^2 = 0.$$

$$t_0 = \sqrt{\varepsilon} \quad \frac{\ln \varepsilon}{1/\sqrt{\varepsilon}}$$

④

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④ We need boundary layer analysis for the following cases.

i) When the small parameter multiplies the highest derivative in the problem. "regular perturbation fails" because the BCs are not all satisfied.

ii) when $\varepsilon \to 0$ changes the character of the problem (in particular in PDE hyperbolic DE → parabolic or Elliptic)

iii) when problems occur in on infinite domains

iv) When singular points are present in the interval of interest

v) model problem has multiple time or length scales.

(5)

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Boundary Layer analysis:

Problem

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0 \qquad 0 < t < 1, \quad 0 < \varepsilon \ll 1$$
$$y(0) = 0, \quad y(1) = 1.$$

a) Inner approximation: Change of variable

$$t = \delta(\varepsilon)\tau, \quad y = Y(\tau)$$

The above DE reduces to

$$y_t = y_\tau \frac{1}{\delta}, \quad y_{tt} = y_{\tau\tau} \frac{1}{\delta^2}$$

$$\Rightarrow \quad \frac{\varepsilon}{\delta^2} y_{\tau\tau} + (1+\varepsilon)\frac{1}{\delta} y_\tau + y = 0$$

$$\frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{\varepsilon}{\delta} y_\tau + \frac{1}{\delta} y_\tau + y = 0$$

we have coefficients $\frac{\varepsilon}{\delta^2}, \frac{\varepsilon}{\delta}, \frac{1}{\delta}, 1$

possible dominant terms: We shall include this term because this term is important in the interior region.

So we have possible cases.

(i) $\frac{\varepsilon}{\delta^2} \sim \frac{\varepsilon}{\delta}$ ii) $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta}$ iii) $\frac{\varepsilon}{\delta^2} \sim 1$

The idea here to find dominant terms. The other terms should be smaller

(i) $\frac{\varepsilon}{\delta^2} \sim \frac{\varepsilon}{\delta}$ $\delta \sim 1$
others are not small.

ii) $\delta \sim \varepsilon$ others are small $\frac{1}{\varepsilon}$ 1

(iii) $\delta = \sqrt{\varepsilon}$ other not small hence we take $\delta(\varepsilon) = \varepsilon$ the.

$\cancel{y_{\tau\tau} + y_\tau}$

$$Y_{\tau\tau} + Y_\tau + \varepsilon Y_\tau + \varepsilon Y = 0 \quad , \quad Y(0) = 0$$

Now regular perturbation is possible

$$Y_0'' + Y_0' = 0 \qquad Y_0(\tau) = C_1 + C_2 e^{-\tau} \; , \; C_2 = -C_1$$

$$y(t) = C_1 (1 - e^{-\tau}).$$

This is the inner solution for $t = O(\varepsilon)$.

$$y(t) = C_1 (1 - e^{-t/\varepsilon})$$

b) <u>outer soln.</u> $t \sim 1$, $y_0' + y_0 = 0$

$$y_0 = c e^{-t} \qquad y_0(1) = 1 \qquad c = e.$$

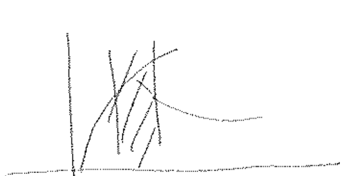
$$y_0(t) = e^{1-t}$$

$$y_0(t) = e^{1-t} \; , \quad y_i(t) = C_1 (1 - e^{-t/\varepsilon}) \qquad t \sim \varepsilon.$$

b) matching: define $t = \sqrt{\varepsilon}\, \eta$.

$$\lim_{\varepsilon \to 0_+} y_i(\eta) = \lim_{\varepsilon \to 0_+} y_0(\eta)$$

$$C_1 = e$$



$t = O(\sqrt{\varepsilon})$ overlap region

$$y_i = e - e^{1-t/\varepsilon}$$

$$y_0 = e^{1-t}$$

c) matching :

$$y_u = y_i + y_0 - e$$

$$= e^{1-t} - e^{1-t/\varepsilon}.$$

y_u satisfies DE.

$y_u(0) = 0,$ $\quad y_u(1) = 1 - e^{1-1/\varepsilon} \;\; = 1 - O(\varepsilon^n)$

$n > 0$

Thus y_u is uniformly valid approxim.

$$\lim_{\varepsilon \to 0^+} \frac{e^{1-1/\varepsilon}}{\varepsilon^n} = 0$$

Another example

$$\varepsilon y'' + y' = 2t \qquad 0 < t < 1, \quad 0 < \varepsilon \ll 1$$

$$y(0) = 1, \quad y(1) = 1$$

2 Another problem

$$\varepsilon y'' + y' + y = 0 \qquad t > 0$$

$$y(0) = 0, \quad \varepsilon y'(0) = 1$$

<u>outer</u> $\quad y' + y = 0 \qquad y = ce^{-t}$

inner

$$t = \varepsilon \tau$$

$$Y_{\tau\tau} + Y_\tau + \varepsilon Y = 0$$

$$Y = c_1 + c_2 e^{-\tau}$$

$$y = c_1 + c_2 e^{-t/\varepsilon} \qquad c_2 = -c_1$$

$$y'(0) = \cdot \frac{1}{\varepsilon} e^{-t/\varepsilon} c_1 \qquad \varepsilon y'(0) = 1$$

$$\Rightarrow c_1 = 1$$

$$y_i(t) = 1 - e^{-t/\varepsilon}, \quad y_0 = ce^{-t}$$

$$\lim_{\varepsilon \to} y_i(\sqrt{\varepsilon}\,\eta) = 1 = \lim_{\varepsilon \to} y_0(\sqrt{\varepsilon}\,\eta) = c.$$

$$y_u = e^{-t} - e^{-t/\varepsilon}$$

<u>verify</u>

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$$r(t,\varepsilon) = \varepsilon\left(e^{-t} - \cancel{\tfrac{1}{\varepsilon^2} e^{-t/\varepsilon}}\right) \cancel{- e^{-t}} + \cancel{\tfrac{1}{\varepsilon} e^{t}}$$
$$\cancel{+ e^{-t}} - e^{-t/\varepsilon}$$

$$= \varepsilon e^{-t} - e^{-t/\varepsilon}$$

Genel

$$\varepsilon y'' + y' p(t) + y\, q(t) = 0$$

<u>outer</u> $$y' + \frac{q(t)}{p(t)} y = 0$$

$$y_0 = C e^{\int_t^1 \frac{q(s)}{p(s)} ds} = b e^{\int_t^1 \cdots}$$

Inner

$$Y_{\tau\tau} + Y_\tau (t) P(\varepsilon\tau) + \varepsilon y q = 0$$

$$\varepsilon = 0 \Rightarrow \quad Y_{\tau\tau} + Y_\tau \cdot p(0) = 0$$

$$Y = C_1 + C_2 e^{-\tau}, \quad y(t) = C_1 + C_2 e^{-t/\varepsilon}$$

$$a = C_1 + C_2 \Rightarrow C_2 = a - C_1$$

$$y_i(t) = C_1 + (a - C_1) e^{-t/\varepsilon}, \quad y_0 = b e^{\int_t^1 \cdots}$$

$$t = \sqrt{\varepsilon}\, \eta$$

$$\lim_{\varepsilon \to 0} y_i(\sqrt{\varepsilon}\,\eta) = C_1 = \underline{b e^{\int_0^1 \cdots}}$$

Example

$$\frac{du}{dt} = -u + (u+\lambda)V$$

$$\varepsilon \frac{dV}{dt} = u - (u+\mu)V$$

$$u(0) = 1, \quad V(0) = 0$$

Here λ and μ are some real constants.

Solution

a) Outer solutions (use regular perturbation).

$u_0 = u_{out}, \quad V_0 = V_{out}$

$$\frac{du_0}{dt} = -u_0 + (u_0+\lambda) V_0$$

$$0 = u_0 - (u_0 + \mu) V_0$$

$$V_0 = \frac{u_0}{u_0+\mu}$$

$$\frac{du_0}{d\tau} = -u_0 + \frac{u_0(u_0+\lambda)}{u_0+\mu} = \frac{u_0(\lambda-\mu)}{u_0+\mu}$$

$$u_0^{\mu} e^{u_0} = C_1 e^{(\lambda-\mu)t} \qquad (1)$$

$du_0(1+\frac{\mu}{u_0})$

$$V_0 = \frac{u_0}{u_0+\mu}$$

$u_0 + \mu \ln u_0 = (\lambda-\mu)t$

$e^{u_0} u_0^{\mu} = e$

(ex2)

b) inner soln: $t = \varepsilon \tau$

$$\frac{du}{d\tau} = \varepsilon\left[-u + (u+\lambda)v\right]$$

$$\frac{dv}{d\tau} = u - (u+\mu)v$$

use regular perturbation method.

$$u_{in} = c_2 \qquad \text{constant}$$

$$\frac{dv_{in}}{d\tau} = c_2 - (c_2+\mu)\, v_{in}.$$

$$\frac{dv_{in}}{d\tau} + (c_2+\mu)\, v_{in} = c_2.$$

$$v_{in}\, e^{(c_2+\mu)\tau} = \frac{c_2}{c_2+\mu}\, e^{(c_2+\mu)\tau} + c_3.$$

$$v_{in} = \frac{c_2}{c_2+\mu} + c_3\, e^{-(c_2+\mu)\cdot t/\varepsilon.}$$

$u_{in}(0) = 1 \Rightarrow c_2 = 1.$

$v_{in}(0) = 0 \qquad c_3 = -\frac{1}{\mu + 1.}$

$$v_{in} = \frac{1}{1+\mu}\left[1 - e^{-(1+\mu)t/\varepsilon}\right]$$

$$u_{in} = 1$$

BCs are used. (ex3)

matching conditions.

i) let $t = \sqrt{\varepsilon}\,\bar{t}$

$$U_{in}(\bar{t}) = U_{out}(\bar{t}) \qquad \text{as } \varepsilon \to 0$$

$t \to 0$

$$1 = U_{out}(0). \qquad U_0 = 1$$

$$u_0^{\lambda} e^{u_0} = C_1 e^{(\lambda-\mu)t}$$

$$\boxed{e = C_1}$$

$$V_{in}(\bar{t}) = V_{out}(\bar{t}) \qquad \text{as } \varepsilon \to 0$$

$$\frac{1}{1+\mu} = \frac{1}{1+\mu} \quad \checkmark$$

ii) unified solution

$$U_u = U_{in} + U_{out} - 1 = U_{out}$$

$$V_u = V_{in} + V_{out} - \frac{1}{1+\mu}.$$

$$= \frac{u_0}{u_0+\mu} - \frac{1}{1+\mu} e^{-(1+\mu)t/\varepsilon}$$

iii) Convergence ✓

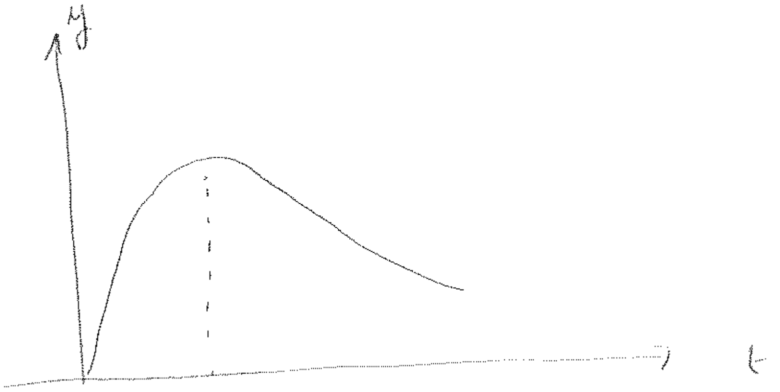
Singular perturbations

Geq

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0 \qquad 0 \le t \le 1, \quad 0 < \varepsilon \ll 1$$
$$y(0) = 0, \quad y(1) = 1$$

exact solution

$$y(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-t} - e^{-t/\varepsilon} \right)$$



$$t_0 = \frac{\varepsilon \ln \varepsilon}{-1+\varepsilon}$$

$$\lim_{\varepsilon \to 0} t_0 = -\frac{1/\varepsilon}{-1/\varepsilon^2} = 0$$

inner solution

$$t = \delta(\varepsilon)\tau \qquad y' = y_\tau \frac{1}{\delta}, \quad y'' = \frac{1}{\delta^2} y_{\tau\tau}$$

$$\frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{1}{\delta}(1+\varepsilon) y_\tau + y = 0$$

$$\frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{1}{\delta} y_\tau + \frac{\varepsilon}{\delta} y_\tau + y = 0$$

Dominant terms:

a) $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta}$ $\qquad \frac{\varepsilon}{\delta}$ and 1 are smaller

$\delta \sim \varepsilon$ ✓

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(Yes) $\Rightarrow \frac{1}{\varepsilon} y_{\tau\tau} + \frac{1}{\varepsilon} y_\tau + y_\tau + y = 0$

$$y_{\tau\tau} + y_\tau + \varepsilon(y_\tau + y) = 0$$

leading order : $y_{\tau\tau} + y_\tau = 0$

$y_\tau + y = c_1 \Rightarrow (e^{\tau} y)_\tau = e^{\tau} c_1$

$y = c_1 + c_2 e^{-\tau}$

or $y(t) = c_1 + c_2 e^{-t/\varepsilon}$, $y(0) = c_1 + c_2 = 0$

$= c_1 - c_1 e^{-t/\varepsilon}$

$y_i(t) = c_1 (1 - e^{-t/\varepsilon})$

<u>outer approximation</u> $t = O(1)$

$y' + y = 0$ $\quad y = ce^{-t}$ $\quad y(1) = 1$, $c = e$

$y_0(t) = e^{1-t}$

matching $\quad y_i \quad t \sim O(\varepsilon)$

$y_0 \quad t \sim O(1)$

overlapping region $\quad t \sim O(\sqrt{\varepsilon})$

$t = \sqrt{\varepsilon}\, \eta$

$$\lim_{\varepsilon \to 0+} y_i(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0+} y_0(\varepsilon \eta)$$

$c_1 = e$

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③

$y_i(t) = e - e^{1-t/\varepsilon}$

$y_o(t) = e^{1-t}$

Uniformly approximated : $y_u = y_i + y_o - e$

$\Rightarrow y_u(t) = e^{1-t} + \cancel{e} - e^{1-t/\varepsilon} - \cancel{e}$

$= e^{1-t} - e^{1-t/\varepsilon}$

6er

$y_u(t) = e(e^{-t} - e^{-t/\varepsilon})$

$y_u(t)$ satisfies the DE exactly.

$y_u(0) = 0$

$y_u(1) = e(e^{-1} - e^{-1/\varepsilon}) = 1 - e^{1-1/\varepsilon}$

$e^{1-1/\varepsilon} \sim O(\varepsilon^n)$

uniformly approximated solution

3 Another example

$$\varepsilon y'' + y' = 2t \qquad 0<t<1 \qquad 0<\varepsilon<<1$$

$$y(0)=1, \quad y(1)=1$$

Inner approximation

$$t = \delta\tau, \quad y' = y_\tau \frac{1}{\delta}, \quad \frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{1}{\delta} y_\tau = 2t$$

$$\delta \sim \varepsilon \qquad y_{\tau\tau} + y_\tau = 2\varepsilon t$$

$$y = c_1 + c_2 e^{-\tau} = c_1 + c_2 e^{-t/\varepsilon}$$

$$y(0) = c_1 + c_2 = 1$$

$$y_i(t) = c_1 + (1-c_1) e^{-t/\varepsilon}$$

Outer approx.

$$y' = 2t, \quad y_0 = t^2 + c$$

$$y(1) = 1 + c = 1 \qquad c = 0$$

$$y_0(t) = t^2$$

matching

$$t = \sqrt{\varepsilon}\, \eta$$

$$\lim_{\varepsilon \to} y_i(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0+} y_0(\sqrt{\varepsilon}\,\eta)$$

$$c_1 = 0$$

$$y_i(t) = e^{-t/\varepsilon}, \quad y_0(t) = t^2$$

uniform approx.

$$y_u(t) = t^2 + e^{-t/\varepsilon}$$

$$r(t) = \varepsilon\left(2 + \frac{1}{\varepsilon^2} e^{-t/\varepsilon}\right) + \cancel{2t} - \frac{1}{\varepsilon} e^{-t/\varepsilon} = \cancel{2t}$$

$$\underline{r(t) = 2\varepsilon}$$

✓

$$y_u(0) = 1, \qquad y_u(1) = 1 + e^{-1/\varepsilon}$$
$$= 1 + O(\varepsilon^n)$$

<u>Uniform approximation</u>:

---

(1) Place of <u>boundary layer</u>. Not always placed at the origin. It may be at the right end point or an interior point.

(2) In practice first assume that an it is placed at the origin. If we meet an inconsistency then change the <u>boundary layer</u>

(3) If the boundary layer is at an other point it is possible to shift it to the origin by

$$\bar{t} = \frac{t_0 - t}{\delta(\varepsilon)}$$

where t_0 is the boundary layer point

(4) One can work also higher order terms we have here studied the leading order analysis

(5) This method is not universal. This means that it can be applied to certain type of problems. See the following theorem

General

Theorem: Consider the boundary value problem

$$\varepsilon y'' + p(t) y' + q(t) y = 0 \qquad 0<t<1, \quad 0<\varepsilon<<1$$
$$y(0)=a, \quad y(1)=b.$$

where p and q are continuous function in $0 \le t \le 1$ and $p(t) > 0$ for $t \in [0,1]$. Then there exist a boundary layer at $t=0$ with inner and outer approximations

$$y_i(t) = C_1 + (a - C_1) e^{-p(0)t/\varepsilon}$$
$$y_0(t) = b e^{\int_t^1 \frac{q(s)}{p(s)} ds}$$
$$C_1 = b e^{\int_0^1 \frac{q(s)}{p(s)} ds}$$

The WKB Approximation:

We shall be interested in the following type equations

$$\varepsilon^2 y'' + q(x) y = 0 \qquad 0 < \varepsilon << 1$$

$$y'' + (\lambda^2 p(x) - q(x)) y = 0 \qquad \lambda >> 1$$

$$y'' + q(\varepsilon x)^2 y = 0 \qquad 0 < \varepsilon << 1$$

Most of these equations come from physics:
Quantum Mechanics: Schrodinger Wave Equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi$$

Seperation of variables $\Psi = \phi(t)\, y(x)$
one dimensional

$$i\hbar \frac{\partial \phi}{\phi \partial t} = \frac{1}{y}\left(-\frac{\hbar^2}{2m} y_{xx} + V y\right) = E$$

$$\phi = c e^{-i E/\hbar\, t}$$

$$-\frac{\hbar^2}{2m} y_{xx} + (V - E) y = 0$$

$$\boxed{\varepsilon^2 y_{xx} + (E - V) y = 0}$$

Let $y = e^{u(x)/\varepsilon}$ — this reminds the singular perturbation (A small discussion on singular perturbation).

$$y' = \frac{1}{\varepsilon} u' e^{u/\varepsilon}$$

$$y'' = \frac{1}{\varepsilon} u'' e^{u/\varepsilon} + \frac{1}{\varepsilon^2} (u')^2 e^{u/\varepsilon}$$

$$\Rightarrow \quad \varepsilon u'' + u'^2 + (E - V) = 0.$$

a) Let $E - V = -k(x)^2$, $v = u'$

$$\varepsilon v' + v^2 - k^2 = 0$$

We now use RPM

$$v(x) = v_0 + \varepsilon v_1 + O(\varepsilon^2)$$

$$\varepsilon v_0' + \varepsilon^2 v_1' + v_0^2 + 2\varepsilon v_0 v_1 - k^2 = 0$$

$$v_0^2 = k^2, \qquad 2 v_0 v_1 = -v_0'$$

$$v_1 = -\frac{v_0'}{2 v_0}$$

$$v = \pm k - \frac{\varepsilon}{2} \frac{v_0'}{v_0} + O(\varepsilon^2)$$

$$= \pm k - \frac{\varepsilon}{2} \frac{k'}{k} + O(\varepsilon^2)$$

$$u = \pm \int_a^x k(x)\,dx - \frac{\varepsilon}{2} \ln k + O(\varepsilon^2)$$

$$y = e^{\pm\frac{1}{\varepsilon}\int_a^x k(x)dx} \frac{1}{\sqrt{k}}$$

$$= \frac{1}{\sqrt{k}} e^{\pm\frac{1}{\varepsilon}\int_a^x k(x)dx}$$

$$\Rightarrow \quad y(x) = \frac{c_1}{\sqrt{k}} e^{\frac{1}{\varepsilon}\int_a^x k(x)dx} + \frac{c_2}{\sqrt{k}} e^{-\frac{1}{\varepsilon}\int_a^x k(x)dx}$$

a to be determined.

Example: $\varepsilon^2 y'' - (1+x)^2 y = 0 \qquad x > 0$

$$y(0) = 1, \quad y(\infty) = 0$$

$$k(x) = 1+x.$$

$$y(x) = \frac{c_1}{\sqrt{1+x}} e^{\frac{1}{\varepsilon}(x+x^2/2)} + \frac{c_2}{\sqrt{1+x}} e^{-\frac{1}{\varepsilon}(x+x^2/2)}$$

$$c_1 = 0 \qquad y(\infty) = 0$$

$$c_2 = 1 \qquad y(0) = 1$$

$$y(x) = \frac{1}{\sqrt{1+x}} e^{-\frac{1}{\varepsilon}(x+x^2/2)}$$

Oscillatory Case : $\varepsilon^2 y'' + k^2 y = 0$

$$y = \frac{c_1}{\sqrt{k}} \cos\left(\frac{1}{\varepsilon}\int_0^x k(x)\,dx\right) + \frac{c_2}{\sqrt{k}} \sin\left(\frac{1}{\varepsilon}\int_0^x k(x)\,dx\right)$$

<u>Example</u> : $y'' + \lambda q(x) y = 0$ $\quad 0 < x < \pi$

$y(0) = y(\pi) = 0$

$k^2 = q(x)\ ,\ \varepsilon = \sqrt{\frac{1}{\lambda}}$

$$y = \frac{c_1}{q^{1/4}} \cos\left(\sqrt{\lambda}\int_a^x \sqrt{q}\,dx\right) + \frac{c_2}{q^{1/4}} \sin\left(\sqrt{\lambda}\int_a^x \sqrt{q}\,dx\right)$$

$a = 0$

$y(0) = 0 \qquad c_1 = 0$

$y(\pi) = 0 \qquad \sqrt{\lambda}\int_0^\pi \sqrt{q}\,dx = n\pi$

$$y_n(x) = \frac{c_2}{q^{1/4}} \sin\left(\frac{\int_0^x \sqrt{q}\,dx}{\int_0^\pi \sqrt{q}\,dx}\, n\pi\right)$$

Ex. $\quad q(x) = (x+\pi)^4$ $\qquad \cancel{\int_0^\pi \sqrt{\pi + x}\,dx = \frac{2}{3}(\pi + x)^{3/2}}$

$\int_0^\pi (x+\pi)^2 dx = \int_\pi^{2\pi} u^2 du = \frac{1}{3}(8\pi^2 - \pi^2)$ $\qquad \cancel{= \frac{2}{3}\, 2^{3/2}\, \pi^{3/2} = \frac{4}{3}\sqrt{2}\,\pi^{3/2}}$

$= \frac{7}{3}\pi^2$ $\qquad \cancel{\frac{4}{3}\sqrt{2}\,\pi^{3/2}\sqrt{\lambda} = n\pi}$

$x + \pi = u$

$\sqrt{\lambda}\,\frac{7}{3}\pi^2 = n\pi$ $\quad \Big|\ \lambda = \frac{9n^2\pi^4}{49}$ $\qquad \cancel{\frac{32}{9}\pi^3 \lambda = n^2\pi}$

$\lambda = \frac{9n^2}{32\pi^2}$

Asymptotic Expansion of Integrals

(1) An example: $y'' + 2\lambda t\, y' = 0$, $y(0) = 0$, $y'(0) = 1$

solution: let $v = y'$ then $v' + 2\lambda t v = 0$

$$v(t) = c e^{-\lambda t^2} \qquad v(0) = 1 \Rightarrow c = 1$$

$$y' = e^{-\lambda t^2} \Rightarrow y(t) = \int_a^t e^{-\lambda s^2} ds.$$

$$y(0) = 0 \Rightarrow a = 0$$

solution $\quad y(t) = \int_0^t e^{-\lambda s^2} ds$

integral representation of a solution: For large values of λ we can approximate this integral.

(2) Laplace Integrals: A type of integral we wish to study is of the form

$$I(\lambda) = \int_a^b f(t)\, e^{-\lambda g(t)} dt, \quad \lambda \gg 1 \qquad (1)$$

where g is a strictly increasing function on $[a,b]$ and the derivative g' is continuous. Here $a < b \leq \infty$ and $\lambda \gg 1$

Eq. (1) can be reduces to

$$I(\lambda) = \int_0^b F(t)\, e^{-\lambda t} dt \qquad \lambda \gg 1.$$

proof: let $s = g(t) - g(a)$ then (1) becomes

$$I(\lambda) = \int_0^{g(b)-g(a)} e^{-\lambda g(a)}\, e^{-\lambda s} f(t(s)) \frac{ds}{g'}$$

$$I(\lambda) = e^{-\lambda g(a)} \int_0^{g(b)-g(a)} F(s) e^{-\lambda s} \, ds \qquad (2)$$

where $F(s) = \dfrac{f(t(s))}{g'(t(s))}$ and $t = t(s)$ is the solution of the equation $s = g(t) - g(a)$

When $b = \infty$ Eq. (2) is the Laplace transform.

The fundamental idea in obtaining an approximation for (2) is to determine what subinterval gives the dominant contribution

Assumptions:

(i) f doesn't grow too fast at infinity
(ii) reasonably well behaved at $t = 0$
(iii) Since $e^{-\lambda t}$ is a rapidly decaying exponential then the main contribution comes from the sub interval about (closer) $t = 0$

(3) Example: Consider the integral

$$I(\lambda) = \int_0^\infty \frac{\sin t}{t} e^{-\lambda t} \, dt \qquad \lambda \gg 1$$

Let T be any positive real number, then

$$I(\lambda) = \int_0^T + \int_T^\infty$$

then consider

$$\left| \int_T^\infty \frac{\sin t}{t} e^{-\lambda t} \, dt \right| \le \int_T^\infty \left| \frac{\sin t}{t} \right| e^{-\lambda t} \, dt$$

$$\le \int_T^\infty e^{-\lambda t} \, dt = \frac{1}{\lambda} e^{-\lambda T}$$

$\left| \frac{\sin t}{t} \right| < 1 \quad t \in [T, \infty)$

hence we call the second integral as "exponentially small term" EST. Hence

$$I(\lambda) = \int_0^T \frac{\sin t}{t} e^{-\lambda t} dt + EST$$

$$= \int_0^T \left(1 - \frac{t^2}{3!} + O(t^4)\right) e^{-\lambda t} dt + EST$$

$$= \frac{1}{\lambda} \int^{\lambda T} \left(1 - \frac{u^2}{6\lambda^2} + O\left(\frac{1}{\lambda^4}\right)\right) e^{-u} du + EST.$$

$$\int_0^\infty u^m e^{-u} du = \Gamma(m+1) = m!$$

hence as $\lambda \to \infty$

$$I(\lambda) = \frac{1}{\lambda} \int_0^\infty \left(1 - \frac{u^2}{6\lambda^2} + O\left(\frac{1}{\lambda^4}\right)\right) e^{-u} du + EST$$

$$\sim \frac{1}{\lambda} - \frac{1}{3\lambda^3} + O\left(\frac{1}{\lambda^5}\right) \qquad \lambda \gg 1.$$

$$\Gamma(x+1) = x\,\Gamma(x) \qquad x > 0$$

for positive integer $\Gamma(n+1) = n!$

$\Gamma(1/2) = \sqrt{\pi}$

<u>Theorem</u>: Consider the integral

$$I(\lambda) = \int_0^b t^\alpha h(t) e^{-\lambda t}\, dt$$

where $\alpha > -1$, $h(t)$ has a Taylor series expansion about $t=0$, with $h(0) \neq 0$, and where $|h(t)| < k e^{ct}$, $0 < t < b$, for some positive ~~integers~~ constants k and c. Then

$$I(\lambda) \sim \sum_{n=0}^{\infty} \frac{h^{(n)}(0)\, \Gamma(\alpha+n+1)}{n!\, \lambda^{\alpha+n+1}}$$

proof:
$$\int_0^b = \int_0^T + \int_T^b$$

$$\left| \int_T^b t^\alpha h(t) e^{-\lambda t}\, dT \right| \leq k \int_T^b |t|^\alpha e^{(c-\lambda)t}\, dt$$

$$\leq k \int_T^b |t|^\alpha e^{-(\lambda-c)t}\, dt$$

$\alpha > -1$

$(\lambda - c)t = s$

$$\leq k \int_{(\lambda-c)T}^{(\lambda-c)b} s^\alpha \frac{1}{(\lambda-c)^{\alpha+1}} e^{-u}\, du$$

$$\leq \frac{k}{(\lambda-c)^{\alpha+1}} \int_{(\lambda-c)T}^{(\lambda-c)b} s^\alpha e^{-u}\, du$$

$$\left| \frac{t}{T} \right| \leq 1 \quad \text{or} \quad \begin{array}{l} |t| \leq b \\ |t| \geq T \end{array}$$

$$\left| \int_T^b t^{\alpha} h(t) e^{-\lambda t} dt \right| \leq k \int_T^b |t|^{\alpha} e^{(c-\lambda)t} dt$$

$$< k|T|^{\alpha} \frac{1}{c-\lambda} \left(e^{(c-\lambda)b} - e^{(c-\lambda)T} \right)$$

$$\leq k|T|^{\alpha} \frac{e^{c}k}{(c-\lambda)} \left[e^{-\lambda b} - e^{-\lambda T} \right]$$

EST.

$\alpha = -1$

$\frac{1}{t}$

hence

$$I(\lambda) = \int_0^T t^{\alpha} h(t) e^{-\lambda t} dt + EST.$$

$$\int_0^T t^{\alpha} h(t) e^{-\lambda t} dt = \sum \frac{1}{n!} h^{(n)}(0) \int_0^T t^{\alpha+n} e^{-\lambda t} dt$$

$$= \sum \frac{h^{(n)}(0)}{n!\,\lambda^{\alpha+n+1}} \int_0^a u^{\alpha+n} e^{-u} du \qquad \lambda t = u$$

$$= \sum \frac{h^{(n)}(0)}{n!} \frac{\Gamma(\alpha+n+1)}{\lambda^{\alpha+n+1}}$$

(4) Example : Error function $\mathrm{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-s^2} ds$

$t = s - \lambda \qquad e^{-(t^2+2\lambda t+\lambda^2)}$

$$\mathrm{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2-2\lambda t-\lambda^2} dt$$

$$= \frac{2}{\sqrt{\pi}} e^{-\lambda^2} \int_0^{\infty} e^{-t^2} e^{-2\lambda t} dt$$

$t = \frac{\tau}{2}$

$$\mathrm{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} e^{-\lambda^2} \int_0^{\infty} e^{-\tau^2/4} e^{-\lambda\tau} \frac{d\tau}{2}$$

$$= \frac{2}{\sqrt{\pi}} e^{-\lambda^2} \left[\frac{1}{2\lambda} - \frac{\Gamma(3)}{(2\lambda)^3} + \frac{\Gamma(5)}{2!\,(2\lambda)^5} - \cdots \right]$$

(5) $\displaystyle\int_{\lambda}^{\infty} e^{-t^2} dt \qquad t^2 = s \qquad t\,dt = \tfrac{1}{2} ds$

$$= \frac{1}{2} \int_{\lambda^2}^{\infty} e^{-s} \frac{1}{\sqrt{s}} ds$$

. Integration by parts gives the same result

Let $I(\lambda) = \int_a^b f(t) e^{\lambda g(t)} dt, \quad \lambda \gg 1$

we have the following

<u>Theorem 2.</u> Let f be continuous function over $[a,b]$, g be sufficiently smooth and has a unique maximum at the point $t = c \in [a,b]$ ($g'(c) = 0$, $g''(c) < 0$). Then

$$I(\lambda) \sim f(c) e^{\lambda g(c)} \sqrt{\frac{-2\pi}{\lambda g''(c)}}, \quad \lambda \gg 1$$

proof: The main contribution comes from where g assumes its maximum.

$$I(\lambda) \sim \int_a^b f(c) e^{\lambda [g(c) + \frac{1}{2} g''(c)(t-c)^2 + \cdots]} dt$$

Letting $s = (t-c)\sqrt{-\lambda g''(c)/2}$ we get

$$I(\lambda) \sim f(c) e^{\lambda g(c)} \sqrt{\frac{-2}{\lambda g''(c)}} \int_{(a-c)\Gamma}^{(b-c)\Gamma} e^{-s^2} ds$$

$$\sim f(c) e^{\lambda g(c)} \sqrt{\frac{-2\pi}{\lambda g''(c)}} \qquad \lambda \gg 1$$

Example: $I(\lambda) = \int_0^\pi e^{\lambda \sin t} dt \qquad t = \pi/2$

$$\sim \int_0^\pi e^{\lambda - \frac{\lambda}{2}(t-\pi/2)^2} dt \sim e^\lambda \int_{-\sqrt{\lambda}\pi/2}^{\sqrt{\lambda}\pi/2} e^{-u^2} \frac{du}{\sqrt{\lambda}} \sim \frac{e^\lambda}{\sqrt{\lambda}} \sqrt{2\pi}$$

$\frac{\sqrt{\lambda}}{\sqrt{2}}(t - \pi/2) = u$

Theorem 3. If the maximum value of $g(t)$ in theorem 2 occurs at one of the endpoints then:

$$I(\lambda) \sim \frac{f(b)e^{\lambda g(b)}}{\lambda g'(b)}, \quad \lambda \gg 1 \quad \text{and } g'(b) \neq 0 > 0$$

$$g'(t) > 0 \quad t \in [a,b]$$

proof: $I(\lambda) = \int_0^b f(t) e^{\lambda g(t)} dt$ — let b be the endpoint where g attains its maximum value

$$\sim \int_0^b f(b) e^{\lambda g(b) + \lambda g'(b)(t-b)} dt$$

$$\sim f(b) e^{\lambda g(b)} \int_0^b e^{\lambda g'(b)(t-b)} dt$$

$$\sim f(b) e^{\lambda g(b)} \int_{\lambda g'(b)(a-b)}^0 e^{-u} \frac{du}{\lambda g'(b)}$$

$$\boxed{\lambda g'(b)(t-b) = -u}$$

$$\sim \frac{f(b)e^{\lambda g(b)}}{\lambda g'(b)} \underbrace{\int_0^\infty e^{-u}}_{1} = \frac{f(b)e^{\lambda g(b)}}{\lambda g'(b)}$$

EX. $$\int_0^1 \sqrt{1+t}\, e^{\lambda(2t-t^2)} dt = \int_0^1 \sqrt{1+t}\, e^{\lambda t(2-t)} dt$$

$t=0$ $$= \int_0^1 \sqrt{1+t}\, e^{(-(t-1)^2+1)\lambda} dt \sim e^{\lambda} \int_0^1 \sqrt{1+t}\, e^{-\lambda(}$$

$\sqrt{\lambda}\,(1-t)=u$

$t=1-\frac{u}{\sqrt{\lambda}}$

$$\sim e^{\lambda}\int_0^1 \sqrt{1+t}\; e^{-\lambda(t-1)^2}\,dt$$

$$\sim e^{\lambda}\int_{\sqrt{\lambda}}^{0} \sqrt{2-\frac{u}{\sqrt{\lambda}}}\; e^{-u^2}\left(-\frac{du}{\sqrt{\lambda}}\right)$$

$$\sim e^{\lambda}\int_0^{\infty} \frac{1}{\sqrt{\lambda}}\sqrt{2-\frac{u}{\sqrt{\lambda}}}\; e^{-u^2}\,du$$

$$\sim e^{\lambda}\int_0^{\infty} \frac{\sqrt{2}}{\sqrt{\lambda}}\left(1-\frac{u}{2\sqrt{\lambda}}+\dots\right) e^{-u^2}\,du$$

$$\sim e^{\lambda}\left[\frac{\sqrt{2}}{\sqrt{\lambda}}\frac{\sqrt{\pi}}{2}-\frac{1}{2\sqrt{2}\,\lambda}+\dots\right] \sim e^{\lambda}\frac{\sqrt{2\pi}}{2\sqrt{\lambda}}+e^{\lambda}O\!\left(\frac{1}{\lambda}\right)$$

<u>from the theorem:</u> $g(t)=2t-t^2$ $\qquad g(0)=0$

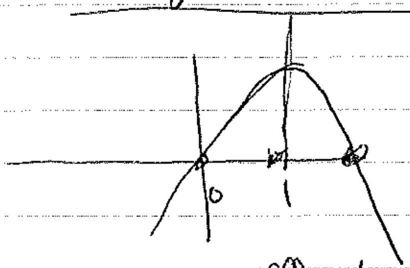
$f(t)=\sqrt{1+t}$ $\qquad t=0$ $\qquad g'(0)=2.$

$g'(t)=2-2t$

$t=1$

$$I(\lambda)=f(0)\frac{e^{\lambda g(0)}}{\lambda g'(0)}\sim\frac{1}{2\lambda}$$

$\boxed{\text{An example where } g'(t)=0 \text{ at the endpoint}}$



$\int_0^{\infty} e^{-u^2}du=\frac{1}{2}\sqrt{\pi}$ $\qquad t=1$ maximum $\qquad g(1)=1$

$g'(1)=0$

$$I(\lambda)=\frac{1}{2}f(1)\,e^{\lambda g(c)}\sqrt{\frac{-2\pi}{\lambda g''(c)}}$$ $\qquad g''(1)=-2$

is one of the endpoint

$f(1) = \sqrt{2}$

$$I(\lambda) = \frac{\sqrt{2}}{2} e^{\lambda} \sqrt{\frac{-2\pi}{\lambda(-2)}} \sim \frac{e^{\lambda}\sqrt{\pi}}{\sqrt{2\lambda}}$$

$$I(\lambda) \sim \frac{e^{\lambda}}{\sqrt{\lambda}} \sqrt{\pi/2}$$

Example: $I(\lambda) = \int_0^{\pi/2} e^{-\lambda \tan^2\theta} \, d\theta$

let $\tan\theta = u$ $\quad d\theta = \cos^2\theta \, du$

$$= \frac{du}{1+u^2}$$

$$I(\lambda) = \int_0^{\infty} e^{-\lambda u^2} \frac{du}{1+u^2}$$

$\sqrt{\lambda}\, u = s$

$$I(\lambda) = \frac{1}{\sqrt{\lambda}} \int_0^{\infty} e^{-s^2} \frac{ds}{1+s^2/\lambda} = \frac{1}{\sqrt{\lambda}} \int_0^{\infty} \left(1 - \frac{s^2}{\lambda} + \frac{s^4}{\lambda^2} - \right) e^{-s^2} d$$

$$= \frac{\sqrt{\pi}}{2\sqrt{\lambda}} - \frac{1}{\lambda\sqrt{\lambda}} \left(\int_0^{\infty} s^2 e^{-s^2} ds \right) \qquad s^2 = y$$

$2s\,ds = dy$

$s^2 e^{-s^2} ds$

$= \ldots$

$= \frac{1}{2} y e^{-y} y^{-1/2} dy$

$= \frac{1}{2} y^{1/2} e^{-y}$

Asymptotic Expansion of Integrals

Consider the differential equations

$$y'' + 2\lambda t y' = 0 \,, \quad y(0) = 0, \quad y'(0) = 1$$

$$\ln y' + \lambda t^2 = A \quad \Rightarrow \quad A = 0$$

$$y' = e^{-\lambda t^2} \quad \Rightarrow \quad y(t) = \int_0^t e^{-\lambda t^2}\, dt$$

Now we determine $y(t)$ for large values of λ.

Laplace Integrals:

Consider the following type of integrals

$$I(\lambda) = \int_a^b f(t)\, e^{-\lambda g(t)}\, dt \qquad \lambda \gg 1$$

where g is a strictly increasing function on $[a,b]$ and the derivative g' is continuous

Actually it is sufficient to examine integrals of the form

$$I(\lambda) = \int_a^b f(t)\, e^{-\lambda t}\, dt \quad , \quad \lambda \gg 1$$

By making $s = g(t) - g(a)$ the first integral reduces to

$$I(\lambda) = \int_a^b f(t) e^{-\lambda g(t)} dt = e^{-\lambda} \int_0^{g(b)-g(a)} \frac{f(s)}{g'(s)} e^{-\lambda s} ds$$

$$= \int_0^{\alpha} F(t) e^{-\lambda t} dt$$

where $F(t) = e^{-\lambda g(a)} \frac{f(t)}{g'(t)}$. Hence

we have

$$I(\lambda) = \int_0^a e^{-\lambda t} f(t)\, dt$$

To find the dominant term. Sub intervals where the <u>integral is dominant</u>.

- $e^{-\lambda t}$ is a rapidly decaying function
- If f doesnot grow too fast at infinity and if f is reasonably well behaved at $t=0$, then it appears that the main contribution comes from a neighborhood $t=0$

"Laplace Method"

<u>Theorem</u>: Consider the integral

$$I(\lambda) = \int_0^b t^\alpha h(t) e^{-\lambda t}\, dt$$

where $\alpha > -1$, where $h(t)$ has Taylor series expansion about $t=0$, with $h(0) \neq 0$ and where $|h(t)| < ke^{ct}$, $0 < t < b$, for some positive constant k and c. Then

$$I(\lambda) \sim \sum \frac{h^{(n)}(0)\, \Gamma(\alpha+n+1)}{n!\, \lambda^{\alpha+n+1}} \quad \text{as} \quad \lambda \to \infty$$

<u>Proof</u>: We split up the integral for any $T > 0$ as

$$I(\lambda) = \int_0^T t^\alpha h(t) e^{-\lambda t}\, dt + \int_T^b t^\alpha h(t) e^{-\lambda t}\, dt$$

The second integral

$$|I_2(\lambda)| \le \int_T^b t^\alpha k e^{ct-\lambda t}\, dt = k \int t^\alpha e^{(c-\lambda)t}\, dt$$

$$\le k \int_T^b t^\alpha e^{(c-\lambda)t}\, dt$$

$$(c-\lambda)t = -u$$

$$\int_T^b \frac{u^\alpha}{(\lambda-c)^\alpha} e^{-u} \frac{du}{\lambda-c} = \frac{1}{(\lambda-c)^\alpha} \int_{-(c-\lambda)T}^{-(c-\lambda)b} u^\alpha e^{-u}\, du$$

$$e^{-\lambda t} < e^{-\lambda T}$$

<u>$\lambda >> 1, \quad t > T$</u>

$$\int_T^b t^\alpha h(t) e^{-\lambda t} \le e^{-\lambda T} \int_T^b |t^\alpha h(t)| \quad \text{EST}$$

hence

$$I(\lambda) = \int_0^T t^{\alpha} h(t) e^{-\lambda t} dt + EST$$

$$h(t) = h(0) + t h'(0) + \cdots$$

$$= \sum_{n=0}' \frac{h^{(n)}(0)\, t^n}{n!}$$

$$I = \sum' \int_0^T \frac{h^{(n)}(0)\, t^{n+\alpha} e^{-\lambda t}}{n!} dt + EST$$

let $\lambda t = u$

$$I(\lambda) = \sum \frac{h^{(n)}(0)}{n!} \int_0^{\lambda T} \frac{1}{\lambda^{n+\alpha+1}} u^{n+\alpha} e^{-u} du + EST$$

$$= \sum' \frac{h^{(n)}(0)}{n!\,\lambda^{n+\alpha+1}} \int_0^{\lambda T} u^{n+\alpha} e^{-u} du + EST$$

as $\lambda \to \infty$

$$I(\lambda) \sim \sum' \frac{h^{(n)}(0)\, \Gamma(n+\alpha+1)}{n!\,\lambda^{n+\alpha+1}}$$

Example $\quad I(\lambda) = \int_0^\infty \frac{\sin t}{t} e^{-\lambda t} dt, \quad \lambda \gg 1$

$$I(\lambda) = \int_0^T \frac{\sin t}{t} e^{-\lambda t} dt + \underbrace{\int_T^\infty \frac{\sin t}{t} e^{-\lambda t} dt}_{I_2(\lambda)}$$

$$|I_2(\lambda)| \leq \int_T^\infty e^{-\lambda t} dt = -\frac{1}{\lambda} e^{-\lambda t} \Big|_T^\infty$$

$$|I_2(\lambda)| \leq \frac{1}{\lambda} e^{-\lambda T} \qquad EST$$

$$I(\lambda) = \int_0^T \frac{\sin t}{t} e^{-\lambda t} dt + EST$$

$$= \int_0^T \frac{1}{t} (t - \frac{1}{5} t^3 + \cdots) e^{-\lambda t} dt + EST$$

$$= \int_0^T (1 - \frac{1}{8} t^2 + \cdots) e^{-\lambda t} dt + EST$$

$$= + \int_0^{\lambda T} \frac{1}{\lambda} (1 - \frac{1}{6} \frac{u^2}{\lambda^2} + \cdots) e^{-u} du$$

as $\lambda \to \infty$

$$I(\lambda) \sim \frac{1}{\lambda} - \frac{2!}{3!\lambda^3} + O(\frac{1}{\lambda^5}), \quad \lambda \gg 1$$

<u>Example</u> error function

$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-s^2} ds.$$

let $s = \lambda + t$

$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2 - 2\lambda t - \lambda^2} dt$$

$$= \frac{2}{\sqrt{\pi}} e^{-\lambda^2} \int_0^{\infty} e^{-t^2} e^{-2\lambda t} dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-\lambda^2} \int_0^{\infty} e^{-\tau^2/4} e^{-\tau\lambda} d\tau$$

Watson lemma applies

$$h(\tau) = e^{-\tau^2/4} \quad , \quad \alpha = 0$$

$$\operatorname{erfc}(\lambda) \sim \frac{1}{\sqrt{\pi}} e^{-\lambda^2} \sum \frac{h^n(0)\, n!}{n!\, \lambda^{n+1}}$$

$$\sim \frac{2}{\sqrt{\pi}} e^{-\lambda^2} \left[\frac{1}{2\lambda} - \frac{2!}{(2\lambda)^3} + \frac{4!}{2!\,(2\lambda)^5} - \frac{6!}{3!\,(2\lambda)^7} + \cdots \right]$$

<u>lemma</u>: let $I(\lambda)=\int_a^b f(t)e^{\lambda g(t)}dt$, $\lambda\gg1$

where f is continous and g is is sufficiently smooth and has a unique maximum at $x=c\in(a,b)$. Then

$$I(\lambda)\sim f(c)e^{\lambda g(c)}\sqrt{\frac{-2\pi}{\lambda g''(c)}}\quad,\quad \lambda\gg1$$

<u>Proof</u>: We expect the main contribution comes from where g assumes its maximum.

To obtain an approximation we replace $f(t)$ by $f(c)$ and replace g by the first tree term in its taylor series

$$I(\lambda)\sim\int_a^b f(c)e^{\lambda(g(c)+\frac{1}{2}g''(c)(t-c)^2)}dt$$

$$\sim f(c)e^{\lambda g(c)}\int_a^b e^{\frac{\lambda}{2}g''(c)(t-c)^2}dt$$

let $s=(t-c)\sqrt{-\lambda g''(c)/2}$ we get

$$I(\lambda)\sim f(c)e^{\lambda g(c)}\sqrt{-\frac{2}{\lambda g''(c)}}\int_{(a-c)\sqrt{-\lambda g''(c)/2}}^{(b-c)\sqrt{-\lambda g''(c)/2}}e^{-s^2}ds$$

as $\lambda \to \infty$

$$I(\lambda) \sim f(c)\, e^{\lambda g(c)} \sqrt{-\frac{2}{\lambda g''(c)}} \underbrace{\int_{-\infty}^{\infty} e^{-s^2} ds}_{\sqrt{\pi}}$$

$$\sim f(c)\, e^{\lambda g(c)} \sqrt{-\frac{2\pi}{\lambda g''(c)}}$$

If the maximum of g occurs at an end point, then $c=a$ or $c=b$ and limit of the integration is zero, thus we will obtain <u>one half value of the above result</u>

<u>example</u>. for $\lambda \gg 1$ approximate the integral

$$I(\lambda) = \int_0^{\pi} e^{\lambda \sin t} dt \sim e^{\lambda} \sqrt{+\frac{2\pi}{\lambda}} \qquad c = \pi/2$$

$$I(\lambda) \sim e^{\lambda} \sqrt{\frac{2\pi}{\lambda}}, \quad \lambda \gg 1$$

Lemma 2 Consider the integral

$$I(\lambda) = \int_a^b f(t)\, e^{\lambda g(t)}\, dt \, , \quad \lambda \gg 1$$

where f is continuous and g is sufficiently smooth and has a unique maximum at $x = b$ with $g'(x) > 0 \quad \forall\, x \in [a, b)$ then prove that

$$I(\lambda) \sim \frac{f(b)\, e^{\lambda g(b)}}{\lambda g'(b)} \, , \qquad \lambda \gg 1$$

Proof.

$$I(\lambda) \sim \int_a^b f(b)\, e^{\lambda [g(b) + (t-b) g'(b)]}\, dt$$

$$\sim f(b)\, e^{\lambda g(b)} \int_a^b e^{\lambda (t-b) g'(b)}\, dt.$$

$$\lambda (t-b) g'(b) = s$$

$$\sim f(b)\, e^{\lambda g(b)} \frac{1}{\lambda g'(b)} \int_{(a-b) g'(b) \lambda}^{0} e^{u}\, du \, .$$

$$\sim f(b)\, e^{\lambda g} \frac{1}{\lambda g'(b)} [$$

If $g'(t) < 0$ for all $t \in [a,b]$ then we have the same result. If the maximum is at $x = a$.

$$I(\lambda) \sim -f(a)\, e^{\lambda g(a)} \frac{1}{\lambda g'(a)} \int_0^{-\lambda(b-a) g'(a)} e^{-s}\, ds$$

$$\sim -f(a)\, e^{\lambda g(a)} \frac{1}{\lambda g'(a)} \left(-e^{-s} \Big|_0^{\infty} \right.$$

$$\sim -f(a) \frac{e^{\lambda g(a)}}{\lambda g'(a)}$$

$$\omega_1 = \tfrac{3}{8}$$

Then (24) becomes

$$u_1'' + u_1 = -\tfrac{1}{4}\cos 3\tau$$

which has the general solution

$$u_1(\tau) = c_1 \cos \tau + c_2 \sin \tau + \tfrac{1}{32}\cos 3\tau$$

The initial conditions on u_1 lead to

$$u_1(\tau) = \tfrac{1}{32}(\cos 3\tau - \cos \tau)$$

Therefore a first-order perturbation solution of (13) is

$$u(\tau) = \cos \tau + \tfrac{1}{32}\varepsilon(\cos 3\tau - \cos \tau) + \cdots$$

where

$$\tau = t + \tfrac{3}{8}\varepsilon t + \cdots$$

This method is successful on a number of similar problems and it is one of a general class of multiple-scale methods (see, e.g., Jordan and Smith [4]).

Generally, the Poincaré–Lindstedt method works successfully on some (not all) equations of the form

$$y'' + \omega_0^2 y = \varepsilon F(t, y, y'), 0 < \epsilon \ll 1$$

These are problems whose leading order is oscillatory with frequency ω_0. The basic technique is to change variables to one with a different frequency, $\tau = (\omega_0 + \omega_1\varepsilon + \ldots)t$, and then assume $y = y(\tau)$ is a perturbation series in ε.

Asymptotics

With insight from the two previous examples we now define some notions regarding convergence and uniformity. It has been observed that substitution of a perturbation series into a differential equation does not always lead to a valid approximate solution. Ideally we would like to say that a few terms in a truncated perturbation series provides, for a given ε, an approximate solution

Theorem: let $I(\lambda) = \int_a^b f(t)\, e^{\lambda g(t)}\, dt$ where

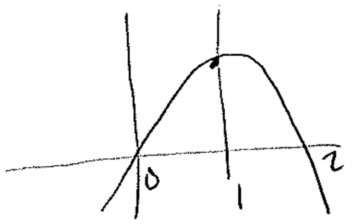
$\lambda \gg 1$, $f(t)$ is a continuous function in $[a,b]$

$g(t)$ is sufficiently smooth and has a unique maximum at $x = c \in [a,b]$. Then the largest term $(g'(c)=0)$.

$$I(\lambda) \sim f(c)\, e^{\lambda g(c)} \sqrt{-\frac{2\pi}{\lambda g''(c)}} \qquad c \in (a,b)$$

$$\sim \frac{1}{2} f(c)\, e^{\lambda g(c)} \sqrt{-\frac{2\pi}{\lambda g''(c)}} \qquad c \text{ is at one of the endpoint}$$

Example: $I(\lambda) = \int_0^1 f(t)\, e^{\lambda(2t-t^2)}\, dt$



$g'(1) = 0$ $\quad$ $t=1$ is also an endpoint

$$I(\lambda) \sim \frac{1}{2} f(1)\, e^{\lambda} \sqrt{\frac{\pi}{\lambda}}$$

Example: $I(\lambda) = \int_0^{\pi} e^{\lambda \sin\theta}\, d\theta$

$g(\theta) = \sin\theta$ $\quad$ $\theta = \pi/2$ is the ext. point

$g''(\pi/2) = -1.$ $\quad$ $I(\lambda) \sim e^{\lambda} \sqrt{\frac{2\pi}{\lambda}}$

Theorem: Let $I(\lambda) = \int_a^b f(t) e^{\lambda g(t)} dt$ where $\lambda \gg 1$, $f(t)$ is a continuous function over $[a,b]$, $g(t)$ is sufficiently smooth and has maximum at one of the end points $x=c$ such that $g'(c) \neq 0$ then the largest value of $I(\lambda)$ is given by

$$I(\lambda) \sim f(b) e^{\lambda g(b)} \frac{1}{\lambda g'(b)} \qquad \text{if } c = b,\ g'(b) > 0$$

$$\sim f(a) e^{\lambda g(a)} \frac{1}{\lambda |g'(a)|} \qquad \text{if } c = a,\ g'(c) < 0.$$

Example: $I(\lambda) = \int_1^2 \sqrt{3+t}\, e^{\frac{\lambda}{t+1}} dt$

$$\sim 2 e^{\frac{\lambda}{2}} \cdot 4/\lambda = \frac{8}{\lambda} e^{\lambda/2}$$

Remark: $u = \frac{1}{t+1}$

$$I(\lambda) = \int_{1/2}^{1/3} \sqrt{\frac{1}{u} + 2}\, e^{\lambda u} \frac{du}{u^2} = -\int_{1/2}^{1/3} \frac{1}{u^{5/2}} \sqrt{1+2u}\, e^{\lambda u}$$

$u = \frac{1}{2} + y$

$$= -8e^{\frac{\lambda}{2}} \int_0^{-1/6} \sqrt{1+y} \cdot \frac{1}{(1+2y)^{5/2}} e^{\lambda y} \sim 8e^{\frac{\lambda}{2}} \int_{-1/6}^{0} e^{\lambda y} \sim 8 e^{\lambda/2} \frac{1}{\lambda}$$

for the entire range of the independent variable t. Unfortunately, as we have seen, this is not always the case. Failure of this regular perturbation method is the rule rather than the exception.

To aid in the analysis of approximate solutions we introduce some basic notation and terminology that permits the comparison of two functions as their common argument approaches some fixed value.

Definition 1.1 *Let $f(\varepsilon)$ and $g(\varepsilon)$ be defined in some neighborhood (or punctured neighborhood) of $\varepsilon = 0$. We write*

$$f(\varepsilon) = o(g(\varepsilon)) \qquad \text{as} \qquad \varepsilon \to 0 \tag{25}$$

if

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0$$

and we write

$$f(\varepsilon) = O(g(\varepsilon)) \qquad \text{as} \qquad \varepsilon \to 0 \tag{26}$$

if there exists a positive constant M such that

$$|f(\varepsilon)| \le M|g(\varepsilon)|$$

for all ε in some neighborhood (punctured neighborhood) of zero.

In this definition $\varepsilon \to 0$ may be replaced by a one-sided limit or by $\varepsilon \to \varepsilon_0$, where ε_0 is any finite or infinite number, with the domain of f and g defined appropriately. If (25) holds, we say f is *little oh* of g as $\varepsilon \to 0$, and if (26) holds we say f is *big oh* of g as $\varepsilon \to 0$. A common comparison function is $g(\varepsilon) = \varepsilon^n$ for some exponent n; another comparison function is $g(\varepsilon) = \varepsilon^n \ln^m \varepsilon$ for exponents m and n.

The statement $f(\varepsilon) = O(1)$ means f is bounded in a neighborhood of $\varepsilon = 0$ and $f(\varepsilon) = o(1)$ means $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. If $f = o(g)$, then f goes to zero faster than g goes to zero as $\varepsilon \to 0$.

Example 1.3 Verify $\varepsilon^2 \ln \varepsilon = o(\varepsilon)$ as $\varepsilon \to 0^+$. By L'Hôpital's rule

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^2 \ln \varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\ln \varepsilon}{(1/\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{(1/\varepsilon)}{(-1/\varepsilon^2)} = 0$$

Example 1.4 Verify $\sin\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0^+$. By the mean value theorem in calculus there is a number c between 0 and ε such that

$$\frac{\sin\varepsilon - \sin 0}{\varepsilon - 0} = \cos c$$

Hence $|\sin\varepsilon| = |\varepsilon\cos c| \le |\varepsilon|$, since $|\cos c| \le 1$. An alternate argument is to note that $(\sin\varepsilon)/\varepsilon \to 1$ as $\varepsilon \to 0^+$. Since the limit exists the function $(\sin\varepsilon)/\varepsilon$ must be bounded for $0 < \varepsilon < \varepsilon_0$, for some ε_0. Therefore $|(\sin\varepsilon)/\varepsilon| \le M$, for some constant M and $\sin\varepsilon = O(\varepsilon)$.

Definition 1.1 may be extended to functions of ε and another variable t lying in an interval I. First we review the notion of uniform convergence. Let $h(t,\varepsilon)$ be a function defined for ε in a neighborhood of $\varepsilon = 0$, possibly not including the value $\varepsilon = 0$ itself, and for t in some interval I, either finite or infinite. We say

$$\lim_{\varepsilon\to 0} h(t,\varepsilon) = 0 \textit{ uniformly on } I$$

if the convergence to zero is at the same rate for each $t \in I$; that is, if for any positive number η there can be chosen a positive number ε_0 independent of t such that $|h(t,\varepsilon)| < \eta$ for all $t \in I$, whenever $|\varepsilon| < \varepsilon_0$. In other words, if $h(t,\varepsilon)$ can be made arbitrarily small over the entire interval I by choosing ε small enough, then the convergence is uniform. If merely $\lim_{\varepsilon\to 0} h(t_0,\varepsilon) = 0$ for each fixed $t_0 \in I$, then we say that the convergence is *pointwise* on I.

One method of proving $\lim_{\varepsilon\to 0} h(t,\varepsilon) = 0$ uniformly on I is to find a function $H(\varepsilon)$ such that the inequality $|h(t,\varepsilon)| \le H(\varepsilon)$ holds for all $t \in I$, and having the property $H(\varepsilon) \to 0$ as $\varepsilon \to 0$. To prove that convergence is not uniform on I it is sufficient to produce a $\bar{t} \in I$ such that $|h(\bar{t},\varepsilon)| \le \eta$ for some positive η, regardless of how small ε is chosen.

2

Definition 1.2 *Let $f(t,\varepsilon)$ and $g(t,\varepsilon)$ be defined for all $t \in I$ and all ε in a (punctured) neighborhood of $\varepsilon = 0$. We write*

$$f(t,\varepsilon) = o(g(t,\varepsilon)) \qquad \textit{as} \qquad \varepsilon \to 0$$

if

$$\lim_{\varepsilon\to 0}\left|\frac{f(t,\varepsilon)}{g(t,\varepsilon)}\right| = 0 \tag{27}$$

pointwise on I. If the limit in (27) is uniform on I, we write $f(t,\varepsilon) = o(g(t,\varepsilon))$ as $\varepsilon \to 0$ uniformly on I. If there exists a positive function $M(t)$ on I such that

$$|f(t,\varepsilon)| \le M(t)|g(t,\varepsilon)|$$

for all $t \in I$ and ε in some neighborhood of zero, then we write

$$f(t,\varepsilon) = O(g(t,\varepsilon)) \qquad as \qquad \varepsilon \to 0, \qquad t \in I$$

If $M(t)$ is a bounded function on I, then we write

$$f(t,\varepsilon) = O(g(t,\varepsilon)) \qquad as \qquad \varepsilon \to 0, \qquad uniformly\ on\ I$$

The big oh and little oh notations permit us to make quantitative statements about the error in a given approximation. In general, the following definition holds.

3

Definition 1.3 *A function $y_a(t,\varepsilon)$ is a uniformly valid asymptotic approximation to a function $y(t,\varepsilon)$ on an interval I as $\varepsilon \to 0$ if the error $E(t,\varepsilon)$ defined by*

$$E(t,\varepsilon) \equiv y(t,\varepsilon) - y_a(t,\varepsilon)$$

converges to zero as $\varepsilon \to 0$ uniformly for $t \in I$.

We often express the fact that $E(t,\varepsilon)$ is little oh or big oh of ε^n (for some n) as $\varepsilon \to 0$ to make an explicit statement regarding the rate that the error goes to zero and whether or not the convergence is uniform.

Example 1.5 Let

$$y(t,\varepsilon) = e^{-t\varepsilon}, \qquad t > 0, \qquad \varepsilon \ll 1$$

The first three terms of the Taylor expansion in powers of ε provide an approximation

$$y_a(t,\varepsilon) = 1 - t\varepsilon + \tfrac{1}{2}t^2\varepsilon^2$$

The error is

$$E(t,\varepsilon) = e^{-t\varepsilon} - 1 + t\varepsilon - \frac{1}{2}\,t^2\varepsilon^2 = -\frac{1}{3!}\,t^3\varepsilon^3 + \cdots$$

For a fixed t the error can be made as small as desired by choosing ε small enough. Thus $E(t,\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$. If ε is fixed, however, regardless of how small, t may be chosen large enough so that the approximation is totally invalid. This phenomenon is illustrated in Fig. 2.3. Thus the approximation is not uniform on $I = [0, \infty)$. Clearly, by choosing $t = 1/\varepsilon$ we have $E(1/\varepsilon, \varepsilon) = e^{-1} - \frac{1}{2}$, which is not small. We may *not* write $E(t, \varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$, uniformly on $[0, \infty)$.

The difficulty of these definitions with regard to differential equations is that the exact solution to the equation is seldom known and thus a direct error estimate cannot be made. Therefore we require some notion of how well an approximate solution satisfies the differential equation and the auxiliary conditions. For definiteness consider the differential equation in (1). We say that an approximate solution $y_a(t, \varepsilon)$ *satisfies the differential equation* (1) *uniformly for* $t \in I$ *as* $\varepsilon \to 0$ if

4 Ref. 4

$$r(t, \varepsilon) \equiv F(t, y_a(t, \varepsilon), \dot{y}_a(t, \varepsilon), \ddot{y}_a(t, \varepsilon), \varepsilon) \to 0$$

uniformly on I as $\varepsilon \to 0$. We can regard $r(t, \varepsilon)$ as the residual error, that is, it measures how well the approximate solution $y_a(t, \varepsilon)$ satisfies the equation.

Example 1.6 Consider the initial value problem

$$\begin{aligned} \ddot{y} + \dot{y}^2 + \varepsilon y = 0, \qquad & t > 0, \qquad 0 < \varepsilon \ll 1 \\ y(0) = 0, \qquad & \dot{y}(0) = 1 \end{aligned}$$

Substituting the perturbation series $y = y_0 + \varepsilon y_1 + \cdots$ gives the initial value problem

$$\begin{aligned} \ddot{y}_0 + \dot{y}_0^2 = 0, \qquad & t > 0 \\ y_0(0) = 0, \qquad & \dot{y}_0(0) = 1 \end{aligned}$$

for the leading-order term y_0. It is easily found that $y_0(t) = \ln(t + 1)$ and hence

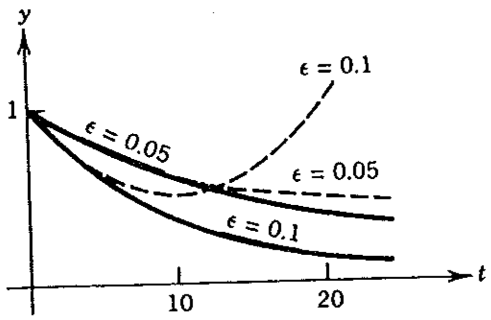


Figure 2.3. Comparison of $y(t, \varepsilon)$ (solid) and $y_a(t, \varepsilon)$ (dashed) for $\varepsilon = 0.1, 0.05$.

$$r(t, \varepsilon) \equiv \ddot{y}_0 + \dot{y}_0^2 + \varepsilon y_0 = \varepsilon \ln(t+1)$$

Thus $r(t, \varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$, but not uniformly on $[0, \infty)$. On any finite interval $[0, T]$, however, we have $|\varepsilon \ln(t+1)| \le \varepsilon \ln(T+1)$, and so $r(t, \varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly on $[0, T]$.

Specifically, the regular perturbation method produces an expansion

$$y_0(t) + y_1(t)\varepsilon + y_2(t)\varepsilon^2 + \cdots$$

for which an approximate solution can be obtained by taking the first few terms. Such an expansion in the integral powers of ε, that is, $1, \varepsilon, \varepsilon^2, \ldots$, is called an *asymptotic power series*. In some problems the expansion is taken to be

$$y_0(t) + y_1(t)\sqrt{\varepsilon} + y_2(t)\varepsilon + y_3(t)\varepsilon^{3/2} + \cdots$$

in terms of the sequence $1, \varepsilon^{1/2}, \varepsilon, \varepsilon^{3/2}, \ldots$. In yet other problems we assume an expansion

$$y_0(t) \ln \varepsilon + y_1(t)\varepsilon \ln \varepsilon + y_2(t)\varepsilon + y_3(t)\varepsilon^2 \ln^2 \varepsilon + y_4(t)\varepsilon^2 \ln \varepsilon + y_5(t)\varepsilon^2 + \cdots$$

The type of expansion depends on the problem. In general we say a sequence $\{g_n(t, \varepsilon)\}$ is an *asymptotic sequence* as $\varepsilon \to 0$, $t \in I$, if

$$g_{n+1}(t, \varepsilon) = o(g_n(t, \varepsilon)) \text{ as } \varepsilon \to 0$$

for $n = 0, 1, 2, \ldots$. That is, each term in the sequence tends to zero faster than its predecessor as $\varepsilon \to 0$. Given a function $y(t, \varepsilon)$ and an asymptotic sequence $\{g_n(t, \varepsilon)\}$ as $\varepsilon \to 0$, the formal series

$$\sum_{n=0}^{\infty} a_n g_n(t, \varepsilon), \qquad a_n \text{ constants} \tag{28}$$

is said to be an *asymptotic expansion* of $y(t, \varepsilon)$ as $\varepsilon \to 0$, if

$$y(t, \varepsilon) - \sum_{n=0}^{N} a_n g_n(t, \varepsilon) = o(g_N(t, \varepsilon)), \qquad \text{as} \qquad \varepsilon \to 0$$

for every N. In other words, for any partial sum the remainder is little oh of the last term. If the limits just cited are uniform for $t \in I$ then we speak of a uniform asymptotic sequence and uniform asymptotic expansion. In most cases the sequence $\{g_n(t, \varepsilon)\}$ is of the form of a product $g_n(t, \varepsilon) = y_n(t)\phi_n(\varepsilon)$ as in the previous examples.

The formal series (28) need not converge. The value of such expansions, although perhaps divergent, is that often only a few terms are required to obtain an accurate approximation, whereas a convergent Taylor series may yield an accurate approximation only if many terms are calculated.

A rather obvious question arises at this point. If a given approximate solution $y_a(t, \varepsilon)$ satisfies the differential equation uniformly for $t \in I$, is it in fact a uniformly valid approximation to the exact solution $y(t, \varepsilon)$? A complete discussion of this question is beyond the scope of this book, but a few remarks are appropriate in order to caution the reader regarding the nature of this problem. Probably more familiar is the situation in linear algebra where we consider a linear system of equations

$$A\mathbf{x} = \mathbf{b}$$

Let $\mathbf{x}_a$ be an approximate solution. A measure of how well it satisfies the system is the magnitude $|\mathbf{r}|$ of the *residual vector* $\mathbf{r}$ defined by

$$\mathbf{r} = A\mathbf{x}_a - \mathbf{b}$$

If $\mathbf{r} = \mathbf{0}$, then $\mathbf{x}_a$ must be the exact solution $\overline{\mathbf{x}}$. But if $|\mathbf{r}|$ is small, it does not necessarily follow that the magnitude $|\mathbf{e}|$ is small, where $\mathbf{e} = \overline{\mathbf{x}} - \mathbf{x}_a$ is the error, or difference between the exact solution and the approximate solution. In *ill-conditioned systems* where $\det A$ is close to zero, a small residual may not imply a small error. A similar state of affairs exists for differential equations. Therefore one must proceed cautiously in interpreting the validity of a perturbation solution. Numerical calculations or the computation of additional correction terms may aid in the interpretation. Often a favorable comparison with experiment leads one to conclude that an approximation is valid.

Exercises

1.1 In a spring-mass problem (see Fig. 2.2) assume that the restoring force is $-ky$ and that there is also a resistive force numerically equal to $a\dot{y}^2$, where k and a are constants with appropriate units. With initial conditions $y(0) = A$, $\dot{y}(0) = 0$ determine the correct time and displacement scales for small damping and show that the problem can be written in dimensionless form as

$$\bar{y}'' + \varepsilon(\bar{y}')^2 + \bar{y} = 0$$
$$\bar{y}(0) = 1, \qquad \bar{y}'(0) = 0$$

where $\varepsilon \equiv aA/m$ is a dimensionless parameter and prime denotes the derivative with respect to the scaled time $\bar{t}$.

1.2 Consider the initial value problem

$$u'' - u = \varepsilon t u, \quad t > 0, \qquad u(0) = 1, \quad u'(0) = -1$$

Find a two-term perturbation approximation for $0 < \varepsilon \ll 1$ and compare it graphically to a six-term Taylor series approximation (centered at $t = 0$) where $\varepsilon = 0.04$. Use a numerical differential equation solver (e.g., in Maple) to find the "exact" solution and compare.

1.3 Verify the following order relations:

(a) $t^2 \tanh t = O(t^2)$ as $t \to \infty$

(b) $\exp(-t) = o(1)$ as $t \to \infty$

(c) $\sqrt{\varepsilon(1-\varepsilon)} = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0^+$

(d) $\dfrac{\sqrt{\varepsilon}}{1 - \cos \varepsilon} = O(\varepsilon^{-3/2})$ as $\varepsilon \to 0^+$

(e) $t = O(t^2)$ as $t \to \infty$

(f) $\exp(\varepsilon) - 1 = O(\varepsilon)$ as $\varepsilon \to 0$

(g) $\displaystyle\int_0^\varepsilon \exp(-x^2)dx = O(\varepsilon)$ as $\varepsilon \to 0^+$

(h) $\exp(\tan \varepsilon) = O(1)$ as $\varepsilon \to 0$

(i) $e^{-\varepsilon} = O(\varepsilon^{-p})$ as $\varepsilon \to \infty$ for all $p > 0$

(j) $\ln \varepsilon = o(\varepsilon^{-p})$ as $\varepsilon \to 0^+$ for all $p > 0$

1.4 Consider the algebraic equation $\phi(x, \varepsilon) = 0$, $0 < \varepsilon \ll 1$, where ϕ is a function having derivatives of all order. Assuming $\phi(x, 0) = 0$ is solvable to obtain x_0, show how to find a three-term perturbation approximation of the form $x = x_0 + x_1\varepsilon + x_2\varepsilon^2$. What condition on ϕ is required to determine x_1 and x_2? Find a three-term approximation to the roots of $\exp(\varepsilon x) = x^2 - 1$.

1.5 Use the Poincaré–Lindstedt method to obtain a two-term perturbation approximation to the following problems:

(a) $y'' + y = \varepsilon y y'^2, \quad y(0) = 1, y'(0) = 0.$

(b) $y'' + 9y = 3\varepsilon y^3, \quad y(0) = 0, y'(0) = 1.$

(c) $y'' + y = \varepsilon y(1 - y'^2), \quad y(0) = 1, y'(0) = 0.$

1.6 To find approximations to the roots of the cubic equation

$$x^3 - 4.001x + 0.002 = 0$$

Why is it easier to examine the equation

$$x^3 - (4 + \varepsilon)x + 2\varepsilon = 0?$$

1.7 Consider the algebraic system

$$0.01x + y = 0.1$$
$$x + 101y = 11$$

In the first equation ignore the apparently small first term $0.01x$ and obtain an approximate solution for the system. Is the approximation a good one? Analyze what went wrong by examining the solution of the system

$$\varepsilon x + y = 0.1$$
$$x + 101y = 11$$

1.8 In Section 1.3 we obtained the initial value problem

$$\frac{d^2h}{dt^2} = -(1 + \varepsilon h)^{-2}, \qquad h(0) = 0, \qquad h'(0) = 1, \qquad 0 < \varepsilon \ll 1$$

governing the motion of a projectile. Use regular perturbation theory to obtain a three-term perturbation approximation. Up to the accuracy of ε^2 terms determine the value t_m when h is maximum. Find $h_{\max} \equiv h(t_m)$ up through ε^2 terms.

1.9 In Exercise 3.9 of Chapter 1 the pendulum problem was scaled to obtain the initial value problem

$$\frac{d^2\theta}{d\tau^2} + \frac{\sin A\theta}{A} = 0, \qquad \tau > 0, \qquad 0 < A \ll 1$$

$$\theta(0) = 1, \qquad \frac{d\theta}{d\tau}(0) = 0$$

Apply the regular perturbation method to find a two-term expansion. Show that the correction term is secular and comment on the validity of the approximation.

1.10 Consider the initial value problem $y'' = \varepsilon t y$, $0 < \varepsilon \ll 1$, $y(0) = 0$,

$0.1x + 10y = 1$

$x + 101y = 11$

$0.1(11 - 101y) + 10y = 1$

$1.1 - 10.1y + 10y = 1$

$-0.1y = -0.1$

$\boxed{y = 1 \quad x = -90}$

$x = x_0 + \varepsilon x_1 + \cdots$

$y = y_0 + \varepsilon y_1 + \cdots$

$\varepsilon x_0 + \varepsilon^2 x_1 + \cdots$

$y_0 + \varepsilon y_1 = 0.1$

$y_0 = 0.1$

$x_0 + y_1 = 0$

$x_0 + 101y_0 = 11$

$x_1 + 101y_1 = 0$

$x_0 = 0.9$

$y_1 = -0.9$

$x_1 = 90.9$

$x = 0.9 + \varepsilon(90.9) + \cdots$

$y = 0.1 + \varepsilon 0.9$

$y'(0) = 1$. Using regular perturbation theory obtain a three-term approximate solution on $t \geq 0$. Does the approximation satisfy the differential equation uniformly on $t \geq 0$ as $\varepsilon \to 0^+$?

1.11 Consider the boundary value problem

$$Ly \equiv t^2\ddot{y} + \varepsilon t^2 \dot{y} + \tfrac{1}{4}y = 0, \qquad 1 \leq t \leq e, \qquad 0 < \varepsilon \ll 1$$
$$y(1) = 1, \qquad y(e) = 0$$

(a) Using regular perturbation find the leading-order behavior $y_0(t)$.

(b) Compute an upper bound for $|Ly_0|$ on $1 \leq t \leq e$ when $\varepsilon = 0.01$. Can you conclude that y_0 is a good approximation to the exact solution?

1.12 Find a two-term perturbation solution of

$$u' + u = \frac{1}{1 + \varepsilon u}, \qquad u(0) = 0, \quad 0 < \varepsilon \ll 1$$

2.2 SINGULAR PERTURBATION

Failure of Regular Perturbation

There are many instances when a straightforward application of the regular perturbation method fails. We do not mean failure for large t due to the appearance of secular terms in the perturbation series as occurred in Example 1.2, but rather a different failure. Often, with a perturbation series some problems do not even permit the calculation of the leading-order behavior because the perturbed problem is of totally different character from the unperturbed problem. The origins of this type of singular behavior will be investigated in the sequel.

The exposition in this section is deliberate and tutorial in nature, and some of the points are subtle and warrant a careful examination. The material herein does not lend itself well to a definition–theorem–proof format, but is instead a discussion that is motivational and intuitive. With these remarks in mind we begin with an example that illustrates the salient points and the inadequacy of a regular perturbation calculation on a finite interval.

Example 2.1 Consider the boundary problem

$$\varepsilon y'' + (1 + \varepsilon)y' + y = 0, \qquad 0 < t < 1, \qquad 0 < \varepsilon \ll 1$$
$$y(0) = 0, \qquad y(1) = 1 \tag{1}$$

Let us assume a perturbation series of the form

SINGULAR PERTURBATIONS: BOUNDARY LAYER ANALYSIS:

Theorem: *Consider the boundary value problem*

$$\begin{aligned} \varepsilon y'' + p(t)y' + q(t)y &= 0, 0 < t < 1, 0 < \varepsilon << 1, \\ y(0) &= a, \quad y(1) = b \end{aligned}$$

where p and q are continuous functions on $0 \leq t \leq 1$ and $p(t) > 0$ for $0 \leq t \leq 1$. Then there exists a boundary layer at $t = 0$ with inner and outer approximations given by

$$y_i(t) = C_1 + (a - C_1)e^{-p(0)\,t/\varepsilon},$$
$$y_o(t) = b\,exp\left(\int_t^1 \frac{q(s)}{p(s)}ds\right)$$

where

$$C_1 = b exp\left(\int_0^1 \frac{q(s)}{p(s)}ds\right)$$

Problems:

(7) Consider the following boundary value problem

$$\begin{aligned} \varepsilon y'' + (1+\varepsilon)y' + y &= 0, \ 0 < t < 1, \ 0 < \varepsilon << 1, \\ y(0) &= 0, \ y(1) = 1 \end{aligned}$$

Use singular perturbation methods to obtain a uniform approximate solution.

(8) Consider the boundary value problems

$$\begin{aligned}
(i) \qquad \varepsilon y'' + y' &= 2t,\ 0 < t < 1,\ 0 < \varepsilon << 1, \\
y(0) &= 1,\ y(1) = 1, \\
(ii)\ \ \varepsilon y'' + y' + y &= 0,\ t > 0, \\
y(0) &= 0,\ \varepsilon\, y'(0) = 1.
\end{aligned}$$

Use the singular perturbation methods to obtain a uniform approximate solution.

(9) Consider the following boundary value problem

$$\begin{aligned}
y' &= -y + (y + \lambda)z, \\
\varepsilon z' &= y - (y + \mu)z, \\
y(0) &= 1, \quad z(0) = 0,
\end{aligned}$$

Use singular perturbation methods to obtain a uniform approximate solution of this system of first order boundary value problem.

(10) Use singular perturbation methods to obtain a uniform approximate solution to the following problems. In each case assume $0 < \varepsilon << 1$ and $0 < t < 1$.

(i) $\varepsilon y'' + 2y' + y = 0, \quad y(0) = 0,\ y(1) = 1,$
(ii) $\varepsilon y'' + y' + y^2 = 0, \quad y(0) = 1/4,\ y(1) = 1/2,$
(iii) $\varepsilon y'' + (1+t)y' = 1, \quad y(0) = 0,\ y(1) = 1 + \ln 2,$
(iv) $\varepsilon y'' + (t+1)y' + y = 0, \quad y(0) = 0,\ y(1) = 1,$
(v) $\varepsilon y'' + t^{1/3}y' + y = 0, \quad y(0) = 0,\ y(1) = exp(-3/2),$
(vi) $\varepsilon y'' + ty' - ty = 0, \quad y(0) = 0,\ y(1) = e,$
(vii) $\varepsilon y'' + 2y' + e^y = 0, \quad y(0) = y(1) = 0,$
(viii) $\varepsilon y'' - (2 - t^2)y = -1, \quad y'(0) = 0,\ y(1) = 1.$

THE WKB APPROXIMATION

The WKB approximation is a perturbation method that we shall apply to the follwong type of boundary value problems

$$\begin{aligned} \varepsilon^2 y'' + q(x)y = 0, \quad 0 < \varepsilon << 1, \\ y'' + (\lambda^2 p(x) + q(x))y = 0, \quad \lambda >> 1, \\ y'' + q(\varepsilon\, x)^2\, y = 0, \quad 0 < \varepsilon << 1 \end{aligned}$$

We sall here consider only the first type of equations. Depneding on the function $q(x)$ we have two different types

Nonoscillatiry Case: $q(x) = -k(x)^2$ where $k(x) > 0$ then the WKB solution takes the form

$$y_{WKB} = \frac{C_1}{\sqrt{k(x)}} exp\left(\frac{1}{\varepsilon}\int_a^x k(\xi)d\xi\right) + \frac{C_2}{\sqrt{k(x)}} exp\left(-\frac{1}{\varepsilon}\int_a^x k(\xi)d\xi\right) \quad (1)$$

Oscillatory Case: $q(x) = k(x)^2$ where $k(x) > 0$ then the WKB solution takes the form

$$y_{WKB} = \frac{C_1}{\sqrt{k(x)}} exp\left(\frac{i}{\varepsilon}\int_a^x k(\xi)d\xi\right) + \frac{C_2}{\sqrt{k(x)}} exp\left(-\frac{i}{\varepsilon}\int_a^x k(\xi)d\xi\right) \quad (2)$$

To find the arbitrary constants C_1 and C_2 we need the boundary conditions. As an illustration consider the following example. Find the WKB solution and eigenvalues λ corresponding to these solutions

$$y'' + \lambda q(x)y = 0,\ 0 < x < \pi, \quad y(0) = y(\pi) = 0,$$

where $q(x) > 0$ for all $x \in [0, \pi]$. This is oscillatory case with $C_2 = 0$

$$\lambda = \lambda_n = \frac{n^2\pi^2}{\left(\int_0^\pi \sqrt{q(\xi)}d\xi\right)^2}$$

for large integer n.

Problems: (11) Find the WKB approximation of the following problem

$$y'' - \lambda(1+x^2)\,y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

for large λ.

(12) Show that the large eigenvalues of the problem

$$y'' + \lambda(x+\pi)^4\,y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

are given by

$$\lambda = \lambda_n = \frac{9n^2}{49\pi^4}$$

for large integers n , and find the corresponding eigenfunctions

(13) The slowly varying oscillator equation is

$$y'' + q(\varepsilon x)^2 y = 0$$

where $\varepsilon << 1$ and q is a strictly positive function. Find and approximate solution. Why is the equation called *slowly varying*?

(14) Find an approximation for the large eigenvalues of

$$y'' + \lambda e^{4x}\,y = 0, \quad 0 < x < 1, \quad y(0) = y(1) = 0.$$

(15) Find an approximation for the general solution of

$$y'' + (\lambda^2 x^2 + x)y = 0, \quad x > 0, \quad \lambda >> 1$$

proceed as in the above oscillatory case by letting

$$y = e^{\frac{iu}{\varepsilon}}, \quad \varepsilon = \frac{1}{\lambda}$$

ASYMPTOTIC EXPANSION OF INTEGRALS

Laplace Integrals

Theorem 2: *Consider the integral*

$$I(\lambda) = \int_0^b t^\alpha \, h(t) \, e^{-\lambda t} \, dt, \tag{3}$$

where $\alpha > -1$, $h(t)$ *has a Taylor series expansion in the neighborhood* $t = 0$, *with* $h(0) ne 0$ *and where* $|h(t)| \le k e^{ct}$ *for some positive constants* k *and* c. *Then (approximately)*

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{h^n(0)\Gamma(\alpha+n+1)}{n!\lambda^{\alpha+n+1}} + EST, \quad \lambda \to \infty \tag{4}$$

where EST *means exponentially small terms.*

This theorem is sometimes called as the Watson's Lemma.

Problems:

(16) Prove that

$$I(\lambda) = \int_0^\infty \frac{\sin t}{t} e^{-\lambda t} \, dt,$$
$$= \frac{1}{\lambda} - \frac{2!}{3!\lambda^3} + O(\frac{1}{\lambda^5}) + EST$$

for large λ ($\lambda >> 1$).

(17) Prove that

$$\frac{2}{\sqrt{2}} \int_\lambda^\infty e^{-t^2} \, dt \sim \frac{2}{\sqrt{2}} e^{-\lambda^2} \left(\frac{1}{2\lambda} - \frac{\Gamma(3)}{(2\lambda)^3} + \frac{\Gamma(5)}{2!(2\lambda)^5} - \cdots \right) \tag{5}$$

Lemma 3: *Let*

$$I(\lambda) = \int_a^b f(t) e^{\lambda g(t)} \, dt$$

where f is continuous and g is sufficiently smooth and has a unuque maximum at a point $x = c$ in (a, b). Then

$$I(\lambda) \sim f(c)e^{\lambda g(c)}\sqrt{\frac{-2\pi}{\lambda g''(c)}}, \quad \lambda >> 1$$

Remark: If the point $x = c$ happens to be at one of the end points a or b then the right hand side of the above expression should be divided by 2.

Lemma 3: *Let*

$$I(\lambda) = \int_a^b f(t)\, e^{\lambda g(t)}\, dt$$

where f is continuous and the maximum of g occurs at $t = b$ one of the end point of the interval with $g'(t) > 0$ for all $t \in [a, b]$. Then

$$I(\lambda) \sim \frac{f(b)\, e^{\lambda g(b)}}{\lambda g'(b)}$$

for $\lambda >> 1$.

Problems : (18) Verify the following approximations for large λ.

(i) $\int_0^\infty e^{-\lambda t} \ln(1 + t^2)dt \sim \frac{2!}{\lambda^3} - \frac{1}{2}\frac{4!}{\lambda^5} + \cdots, \quad \lambda >> 1,$
(ii) $\int_0^1 \sqrt{1+t}\, e^{\lambda(2t-t^2)}\, dt \sim \sqrt{\frac{\pi}{2\lambda}}\, e^{\lambda}. \quad \lambda >> 1,$
(iii) $\int_1^2 \sqrt{3+t}\, e^{\frac{\lambda}{t+1}} dt \sim \frac{8}{\lambda}e^{\lambda/2}, \quad \lambda >> 1$

(19) Find a two term approximation for large λ for

$$\int_0^{\pi/2} e^{-\lambda \tan^2\theta}\, d\theta$$